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Monetary Policy Implementation with Ample Reserves

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Abstract: We offer a parsimonious model of the reserve demand to study the tradeoffs associated with various monetary policy implementation frameworks. Prior to the 2007–09 financial crisis, many central banks supplied scarce reserves to execute their interest-rate policies. In response to the crisis, central banks undertook quantitative-easing policies that greatly expanded their balance sheets and, by extension, the amount of reserves they supplied. When the crisis and its aftereffects passed, central banks were in a position to choose a framework that has reserves that are (1) abundant—by keeping their balance sheets and reserves at the expanded level; (2) scarce—by vastly decreasing their balance sheets and reserves; or (3) somewhere in between abundant and scarce—by moderately decreasing their balance sheets and reserves. We find that the best policy implementation outcomes are realized when reserves are somewhere between scarce and abundant. This outcome is consistent with the Federal Open Market Committee's 2019 announcement to implement monetary policy in a regime with an ample supply of reserves.

JEL classification: E42, E58

Key words: federal funds market, monetary policy implementation, ample reserves

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1 Introduction

A monetary policy implementation framework describes the targets and tools a central bank uses to transmit its desired stance of monetary policy to financial markets and the real economy. An implementation framework specifies, e.g., a target interest rate, reserve requirements, the rate of remuneration on reserves, the discount rate, tools—such as open market operations—used to adjust the quantity of reserves, parameters associated with standing repo and reverse repo facilities, the issuance of central bank bills and so on. Over time and across jurisdictions central banks have chosen a wide variety of monetary implementation schemes. Although monetary economics typically abstracts from implementation issues by assuming that the central bank achieves its policy stance by choosing *the* policy interest rate, such an assumption is not at all innocuous. In practice, the way in which monetary policy is implemented can constrain the choice of feasible policy stances and, conversely, the choice of a policy stance has implications for how the policy should be implemented. This paper attempts to shed some light on what constitutes an optimal implementation framework.

The 2007-2009 financial crisis and its aftermath highlight some of the interactions between a policy stance and implementation strategies. A conventional policy response at the outset of the financial crisis would call for negative nominal policy rates. However, it is challenging to implement non-negligible negative nominal policy rates in an economy with physical currency. As a result, many central banks adopted new policy tools, such as forward guidance and large-scale purchases of long-dated assets—quantitative easing—to provide additional stimulus. While large-scale asset purchases were effective at easing financial conditions, they also dramatically increased the amount of reserves supplied to the banking system. The pre-crisis tools for controlling overnight interest rates—such as open market operations via overnight repo and reverse-repo transactions—became ineffective since open market operations that result in small changes in the supply of reserves cannot affect overnight rates since reserves are no longer scarce. In response to the impotence of traditional tools, central banks introduced new and additional ones, such as payment of interest on reserves and overnight repurchase and reverse repurchase facilities, to better control the policy rate (Bernanke (2020); Ihrig, Meade, and Weinbach (2015)).

The framework for monetary policy implementation continues to evolve to this day. In the months prior to the coronavirus pandemic, central banks around the world were unwinding their responses to the financial crisis and "normalizing" their policy stances. However, the disruptions in money markets in September 2019 and those associated with the Covid-19 pandemic stalled the normalization process and led policymakers to once again reassess their implementation frameworks. It is likely that lessons learned since September 2019 will shape the narrative for a monetary policy implementation framework into the foreseeable future.

In this paper, we develop a simple model of the banking system's demand for reserves to better understand the choices policy makers face when selecting a monetary policy implementation framework. Throughout we focus on the experience of the Federal Reserve to create an intuitive link between theory and practice. Our model builds on the seminal work of Poole (1968): a model where banks hold reserves to meet reserve requirements, borrow or lend them in an interbank market to adjust end-of-day reserves and face a late-period payment shock that can drain reserves after the interbank market closes.[1] We generalize the Poole (1968) model along several dimensions. First, we consider a reserve demand function that captures banks' preferences for reserves in the post-crisis world beyond required reserves. Second, we include shocks to banks' reserve demand functions that reflect the increased uncertainty associated with estimating the banking system's demand for reserves in the post-crisis period. And finally, we introduce shocks to supply of reserves to incorporate uncertainty that arises from factors outside the Federal Reserve's control, such as changes in the balances that the Treasury Department holds at the Federal Reserve or in the balances held at the overnight reverse repo facility, both of which have become rather pronounced and important in the post-crisis period.

Our model generates a downward sloping reserve demand curve with three main regions—a region of "high" aggregate reserves, a region of "low" aggregate reserves and a smooth transition between the two—that is consistent with the demand curve estimated by Afonso et al. (2023). Through the lens of our model, we define the amount of reserves that a central bank supplies to the banking system as abundant, scarce or ample. Reserves are *abundant* when the equilibrium is characterized by no interest rate volatility.[2] Such an equilibrium occurs only if the central bank supplies very large quantities of reserves. In Figure 1, the demand curve is flat when aggregate reserves are abundant since banks are able to meet

[1] Many models examining some aspect of monetary policy implementation use the Poole model as their starting point.

[2] In our model, no interest rate volatility means that the probability that the policy rate deviates from its target is very small.

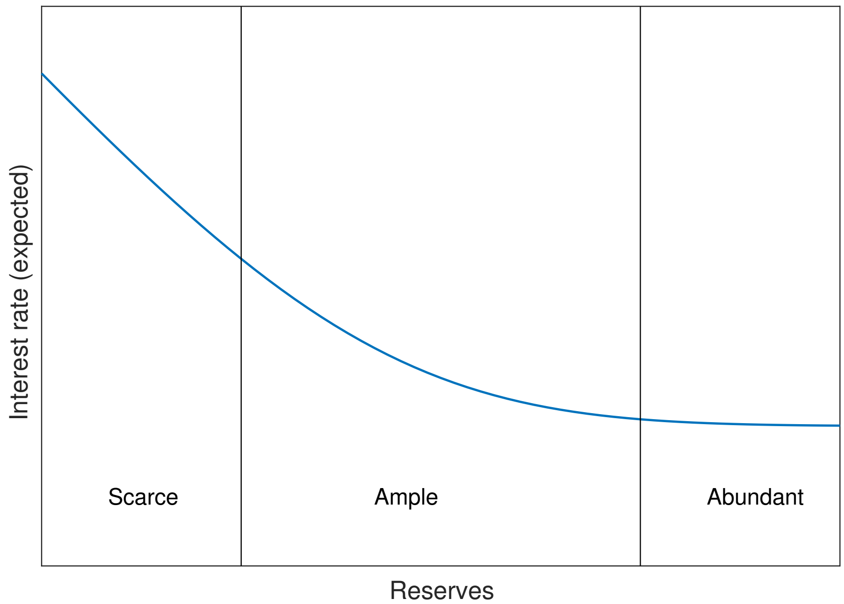

Figure 1: Reserve Demand

their internal and external regulatory requirements for any (ex post) reserve demand shock or supply shock realizations. When reserves are abundant, the value of trading reserves is simply the interest paid on overnight reserve balances. In contrast, reserves are *scarce* when the equilibrium displays high-rate volatility, even when the central bank undertakes open market operations in an attempt to stabilize rates. Intuitively, this equilibrium occurs when reserve supply is small and the equilibrium rate is on the inelastic downward-sloping part of the demand curve, as illustrated in Figure 1. In the region of scarce reserves, the marginal value of reserves increases as aggregate reserves decline and always exceed the interest paid on overnight reserves. Finally, in between scarcity and abundance, we define reserves to be *ample*. In this region, reserve supply and demand shocks result in a positive but suppressed equilibrium interest rate volatility.[3]

We use our model to study the choice of a central bank's monetary policy implementation framework. In practice, policy makers have preferences over outcomes and operations and, because of this, they face trade-offs when choosing their implementation framework. For example, policy makers may prefer low, rather than high, volatility in their policy rate. They may also prefer smaller, less frequent open market operations in response to reserve

[3]These definitions—scarce, abundant, and ample—are our own, and are intended to facilitate the discussion of central banks' implementation choices from within the perspective of our model.

supply and demand shocks to larger or frequent operations. A central bank could achieve a low volatility policy rate, without relying on open market operations, by supplying a very large amount of reserves to the banking sector. But, if policy makers also prefer smaller to larger balance sheets, then a large balance sheet—that stabilizes the policy rate—may not necessarily constitute an optimal framework from the central bank's perspective. We capture these policy-maker preferences as a linear combination of costs associated with: (i) volatility in the policy rate; (ii) the size of the central bank's balance sheet; and (iii) the expected size of open market operations. The central bank's implementation framework specifies the initial quantity of aggregate reserves—small, moderate, or large—and the size and frequency of subsequent open market operations. The *optimal* implementation framework is the one that minimizes a linear combination of these costs.

Since the 2007-2009 financial crisis, the magnitudes of the shocks to the reserve supply have increased substantially and a new set of drivers of reserve demand have also emerged, both of which decrease the predictability of reserve supply and demand.[4] We show that in the post-financial crisis world the optimal monetary policy framework has aggregate reserve supply in the intermediate region, between scarcity and abundance. These findings are consistent with the Federal Reserve's plans to implement monetary policy over the longer run in an environment of ample reserves.[5] An important implication of this framework is that high frequency, active adjustment of the reserve supply is not needed to implement policy, although occasional adjustments may arise.[6]

The next section provides a brief overview of the banking system's demand for reserves in the pre- and post-financial crisis periods, as well as a discussion of reserve supply in the post-crisis period. Section 3 introduces our model of a monetary policy implementation framework

[4]Post-crisis liquidity regulations, such as the Liquidity Coverage Ratio (LCR), living wills, stress testing, as well as banks' responses to the regulation via internal liquidity management strategies and targets, have transformed the demand for reserves. In recent years, changes in reserve supply due to factors outside of the control of the central bank—mainly, balances in the account of the U.S. Treasury or at the overnight reserve repo facility—have increased too, making reserve supply more uncertain.

[5]In January 2019, the Federal Open Market Committee (FOMC) announced its intention to maintain an "ample supply of reserves" and to use administered interest rates, such as the rate paid on reserves, as its primary tools to ensure rate control. Our model predictions are also consistent with the Federal Reserve's longstanding plan to operate with a balance sheet that is no larger than necessary for efficient and effective policy implementation, see "Policy Normalization Principles and Plans," September 2014, available at https://www.federalreserve.gov/monetarypolicy/policy-normalization-discussions-communications-history.htm.

[6]See "Statement Regarding Monetary Policy Implementation and Balance Sheet Normalization." https://www.federalreserve.gov/newsevents/pressreleases/monetary20190130c.htm.

and provides insights into a central bank's choice of the optimal framework. Section 4 then focuses on how effective these regimes have been in practice and discusses potential financial stability implications of regimes with high reserves. Section 5 concludes.

2 Reserve Demand and Supply

To motivate our model and analysis, we first discuss the impact that post-crisis liquidity regulations have had on the banking sector's demand for reserves and how these regulations, along with other considerations, make it more challenging for a central bank to estimate total reserve demand. We then document changes in the supply of reserves for the pre- and post-crisis periods in the U.S. and show that reserve supply volatility has substantially increased in the post-financial crisis period.

2.1 Reserve Demand

Prior to the 2007-2009 financial crisis, the Federal Reserve and clearing banks provided intraday liquidity at generous terms so that banks could almost costlessly smooth out their daily payment flow obligations. As a result, banks demanded reserves mainly to satisfy their end-of-day reserve requirements. Since the primary driver for pre-crisis reserve demand was banks' reserve requirements, the Federal Reserve was able to estimate banks' total demand for reserves with a high degree of precision.

Post-crisis regulations have directly and indirectly affected banks' liquidity risk management in ways that have resulted in new and more uncertain sources of demand for reserves. For example, the liquidity coverage ratio (LCR) requires banks to hold a sufficient amount of high-quality liquid assets (HQLA) to meet net cash outflows over a thirty-day stress period. HQLA include central bank reserves and government securities, as well as some other safe and liquid assets. The LCR regulation implies that banks' demand for the sum of reserves *and* government securities will be higher than the pre-crisis period. Banks are, however, free to allocate their HQLA holdings between government securities and reserves as they see fit: the LCR does not, *per se*, specify any requirements about reserve holdings *vis-à-vis* government securities holdings. Since the LCR allows banks to choose different mixes of reserves and other types of HQLA to satisfy the requirements, the Federal Reserve may be unable to estimate banks' total demand for reserves with a high degree of precision, even though banks' demand for HQLA can be. For example, a bank may prefer to hold more government securities than reserves if the yields on government securities are relatively high but may suddenly reverse this preference if the bank believes that it might be challenging to quickly convert government securities into cash in the face of outflows.

In addition to the increase in demand for reserves associated with regulatory requirements such as the LCR, resolution plans under the Dodd-Frank Act, buffers of highly liquid assets under Regulation YY, and banks' internal liquidity stress tests also increase banks' demand for reserves. Importantly, the impact on the demand for reserves depends on banks' own risk assessments and their willingness to bear these risks. Owing to these new and more complex sources of demand for bank reserves, a central bank's ability to accurately predict reserve demand has been reduced in the post-financial crisis period.

In Section 3, we model the increased uncertainty regarding banks' reserve demand in the post-crisis period as an increase in the magnitude of shocks to reserve demand and as a decrease in the central bank's ability to predict the reserve demand.

2.2 Reserve Supply

Traditional models typically assume that the reserve supply is under the complete control of the central bank. In practice, however, the level of reserves can change due to factors that are outside of the control of the central bank, the so-called autonomous factors. Two important examples of these factors in the U.S. are the balances at the overnight reverse repo (ON RRP) facility and the balances that the Treasury Department holds at the Federal Reserve. In this section, we show that variations in reserve supply due to autonomous factors have become much larger in the U.S. in recent years. This development poses challenges for monetary policy implementation when reserves are scarce and small changes in reserves affect the policy rate.

In the absence of offsetting open market operations, reserves available to banks change on a daily basis. Reserves may change for two reasons: First, the size of the central bank balance sheet may change as the result of, e.g., large-scale asset purchases. Second, the composition of the central bank liabilities may change when, e.g., bank reserves are converted into physical currency, which is also a liability of the central bank, or when currency is returned to the central bank and the returning bank's account at the central bank is credited (with reserves). Another example of changes in reserves occurs on tax payment dates when funds from a bank's account at the central bank are used to pay taxes, which reduces reserves in the banking system and increases the balance of the Treasury General Account (TGA) at the central bank. Prior to the 2007-2009 financial crisis, these exogenous, day-to-day changes in the supply of reserves were small and mostly predictable. Since then, the volatility of these changes has increased significantly.

The first 5 panels in Figure 2 illustrate the weekly volatility of selected autonomous factors from 2003 to 2022. Notice that the volatility of these factors have significantly increased since 2009. This, in turn, means that the volatility of the supply of reserves, the bottom-right panel in Figure 2, has also increased substantially. The volatility of the reserve supply was close to zero prior to the 2007-2009 financial crisis, partly because the Federal Reserve actively offset movements in the supply of reserves. However, given the much smaller volatility of autonomous factors in the pre-crisis period, the reserve volatility would have been much lower than it has since 2009 even in the absence of the Federal Reserve's operations.

The Federal Reserve could, in principle, reduce the volatility of autonomous factors, at least in a limited way.[7] It is not, however, obvious that reducing reserve volatility is desirable. For example, since 2015 the Treasury Department has tried to maintain a five-day liquidity buffer in its account at the Federal Reserve to limit the risk that it may be unable to access markets due to an operational outage or a cyber-attack.[8] While such a buffer contributes to a larger autonomous factor, it might not be possible or desirable to return to the pre-crisis balance. Alternatively, the Treasury Department could move its buffer into the banking sector, thereby reducing or eliminating this autonomous factor. But banks may not be interested in taking on large and volatile cash deposits since it is costly from both a

[7]It would be difficult to regulate the withdrawal and deposit of physical currency. The Federal Reserve could set up rules on the use of accounts held by non-banks. But it would be difficult to force these accounts to substantially reduce their volatility without impairing their operational needs. In fact, free withdrawal and deposit is a primary advantage of holding cash or reserves.

[8]Treasury's May 6, 2015, quarterly refunding statement notes: "Based on our review, the TBAC's [Treasury Borrowing Advisory Committee's] recommendations, and an assessment of emerging threats, such as potential cyber-attacks, Treasury believes it is prudent to change its cash management policy starting this month. To help protect against a potential interruption in market access, Treasury will hold a level of cash generally sufficient to cover one week of outflows in the Treasury General Account, subject to a minimum balance of roughly $150 billion."

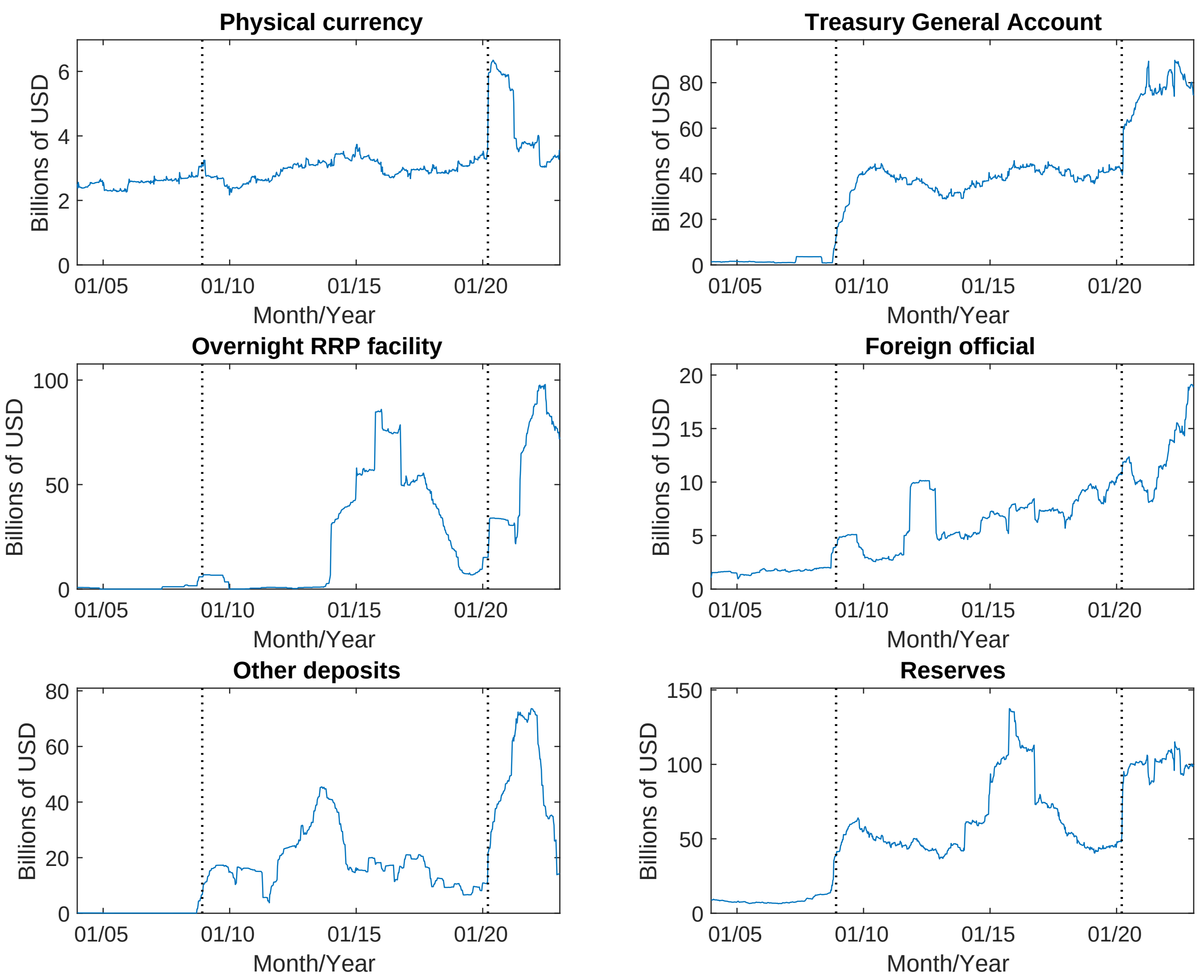

Figure 2: Volatility in Selected Autonomous Factors and Reserves in the United States

The volatility of autonomous factors and reserves generally increased since 2009, albeit not monotonically. Time period covered is 2003 to 2022. Volatility is calculated as the standard deviation of weekly differences over a 52-week trailing window, using publicly released weekly snapshots of Federal Reserve's liability (H.4.1 releases from the Federal Reserve). Vertical lines mark the beginning of asset purchases in response to the 2007-09 financial crisis in late 2008 and to the Covid-19 pandemic in early 2020. Other deposits are held by selected official and private entities.

regulatory and liquidity risk management perspective.

As illustrated in the bottom-right chart on Figure 2, the weekly volatility of reserves was less than $10 billion prior to the financial crisis. Volatility increased sharply during the first round of large-scale asset purchases to around $60 billion in 2009. For several years after the crisis, reserve supply volatility remained near that level through different rounds of large-scale asset purchases. Between 2015 and 2017, a period between the end of asset purchases and the beginning of balance sheet normalization, reserve volatility was elevated and reached nearly $150 billion. During this period, volatility reflected changes in reserves due to exogenous factors as the Federal Reserve no longer conducted daily open market operations to fine-tune reserve supply nor changed the size of its balance sheet to conduct monetary policy.[9] Volatility fluctuated around $50 billion during balance sheet normalization over 2017-2019, and increased again as the Federal Reserve expanded the balance sheet in response to the Covid-19 pandemic.

3 A Model of a Monetary Policy Implementation Framework

We propose a model of interbank interest rate determination to study the trade-offs that policy makers face when choosing a monetary policy framework. Our model builds on the work of Poole (1968) and accommodates implementation frameworks with scarce, abundant, and somewhere in between (ample) reserves. As in Poole (1968), banks must hold a minimum level of reserves. In contrast to Poole (1968), where the minimum level is proportional to bank deposits, in our model, the minimum level of reserves captures banks' expanded demand for reserves in a post-financial crisis environment. More specifically, and as discussed in Section 2, changes in (i) liquidity regulation, (ii) supervision of banks, (iii) banks' risk management practices and (iv) the structure of the market for reserves have substantially transformed—and expanded—the banking system's demand for reserves relative to the pre-crisis period. These new and more complex sources of demand for reserves have reduced the

[9]Fine-tuning the reserve supply was a key element of the monetary policy implementation framework in the pre-crisis period. Therefore, pre-crisis figures do not reflect volatility in exogenous reserve supply since most of the autonomous changes were "reversed" through open market operations. Similarly, reserve-injecting operations in the aftermath of mid-September 2019 money market volatility tended to offset reductions in reserve supply.

central bank's ability to accurately predict demand for reserves. Our model captures this increased uncertainty, which is an important feature of the post-financial crisis period, by introducing shocks to the demand for reserves. These demand shocks decrease the central bank's ability to accurately predict aggregate reserve demand. Moreover, the volatility of factors outside the control of a central bank that affect the supply of reserves has also increased relative to the pre-financial crisis period, as illustrated in Figure 2. We model the increased uncertainty arising from these autonomous factors by introducing shocks to the supply of reserves.[10]

3.1 Agents

There are two types of agents: depository institutions, which we will refer to as "banks," and a central bank.

Banks. There are N banks in the banking system indexed by $i \in \{1, ..., N\}$. Banks are risk-neutral and maximize expected profits. Each bank has an initial level of desired reserves, $\bar{R}_i$, which is exogenous to the model. Each bank receives a demand shock, d_i, that changes the bank's desired reserves by d_i, where d_i can be either positive or negative. After the demand shock realizations, banks trade in a competitive interbank market for reserves, e.g., the federal funds market in the U.S., to adjust their reserve holdings at rate r. After the interbank market closes, each bank receives a late (Poole) shock, u_i, which redistributes reserves between banks. If this shock causes reserves to fall below $\bar{R}_i + d_i$, then bank i borrows reserves from the central bank at a penalty rate, r_P, to get reserves back to its desired level $\bar{R}_i + d_i$. One can think of the penalty rate as the rate charged by the central bank on discount window loans or the minimum bid rate at the Standing Repo facility . Banks hold all of their reserves overnight at the central bank and earn the interest rate on these reserves equal to $r_{IOR} < r_P$.

Central bank. The central bank chooses an initial level of reserves to supply to the banking system, R. The supply of reserves is subject to a shock, s, which arises from factors outside the central bank's control. After the shock, the central bank either injects, $x > 0$, or drains, $x < 0$, reserves through an open market operation. The central bank lends to banks at a

[10] Shocks in Poole (1968) are "late-period" shocks, which redistribute reserves between banks after the interbank market closes. As we discussed in Section 3.2, our model includes this late-period shock, and incorporates shocks in the demand for and supply of reserves.

penalty rate, r_P, and remunerates reserve balances that banks hold at the central bank at the rate r_{IOR}.

3.2 Shocks

There are three important shocks at play: a supply shock, a demand shock, and a late (Poole) reserve redistribution shock.

Supply shock. The shock s to the supply of reserves captures that central banks do not perfectly control the supply of reserves. Changes in the balances at the ON RRP facility or at the account that the Treasury Department holds at the Federal Reserve are relevant examples of supply shocks that affect the level of reserves in the U.S. banking system.

Demand shock. The model incorporates a shock, d_i, to a bank's initial demand for reserves, where $d = \sum_i d_i$ denotes the aggregate shock to the banking system demand for reserves. This shock captures the ex ante uncertainty about a bank's demand for desired reserves, as well as the increased uncertainty associated with the post-financial crisis environment.

Late shock. Each bank receives a late shock, u_i. This shock represents increases or decreases in a bank's reserves due to, e.g., a payment that occurs late in the day after the interbank market closes. We can interpret the u_i shocks as a reshuffling of existing reserves among the N banks, where $\sum_i u_i = 0$. Late shocks will further have the indirect effect of increasing the supply of reserves in the system if banks borrow from the central bank at the end of the day.

3.3 Timing of Events

The model is static in the sense that events and actions occur over a finite number of periods, six to be precise. We think of the model as describing what happens over a typical day. The timing of events is as follows:

> **Time $\mathbf{t = 1}$:** The central bank: (i) chooses the level of reserves, R, to supply to the banking system; (ii) specifies the penalty rate, r_P, at which banks can borrow from the central bank; and (iii) sets the interest paid on reserves, r_{IOR}. Each bank has an initial (time $t = 1$) level of desired reserves, $\bar{R}_i$, where $\bar{R} = \sum_i \bar{R}_i$.
>
> **Time $\mathbf{t = 2}$:** The reserve supply shock, s, is revealed. Total reserves are now $R + s$.

Time t = 3: The central bank either injects $x \geq 0$ or drains $x \leq 0$ reserves. Total, and final, supply of reserves is $R + s + x$.

Time t = 4: Each bank receives a demand shock, d_i, to its initial level of desired reserves. Bank i's desired reserves are now given by $\bar{R}_i + d_i$. The banking system's demand for reserves is $\bar{R} + d$, where $d = \sum_i d_i$.

Time t = 5: A competitive interbank market opens, where banks can borrow and lend reserves at rate r. Reserves are redistributed in the interbank market, and then the market closes. Denote bank i's reserves at the end of period 5 by R_i.

Time t = 6: Each bank i receives a late shock, u_i, to its reserve holdings. If $R_i + u_i < \bar{R}_i + d_i$, then bank i must borrow the difference from the central bank at rate r_P. Banks hold their reserves at the central bank and earn interest r_{IOR}.

The central bank makes decisions about the level of reserves at $t = 1$, when it chooses the initial supply of reserves, and at $t = 3$, when it adjusts this initial level plus the reserve supply shock through an open market operation. When making these decisions, the central bank takes into account banks' expected borrowing and lending in the interbank market at $t = 5$. Banks' behavior creates a demand for reserves which, along with the central bank's earlier supply decisions, results in an equilibrium interbank (fed funds) rate at $t = 5$. Intuitively, if the central bank cares only about rate control, and there are no costs associated with the size of its balance sheet or with open market operations, then the solution to the central bank's problem is relatively simple: the central bank provides an arbitrarily large supply of reserves—in the abundant region—so that the reserve supply intersects the reserve demand in the far right region where the demand curve is essentially flat in Figure 1. For a large enough level of reserves, the equilibrium interbank rate will (almost always) equal the rate on overnight reserves, r_{IOR}, regardless of the supply, s, and demand, d, shock realizations. In this scenario, the central bank is able to (almost) perfectly control its policy rate with an abundant reserve supply.[11] However, if there are costs associated with the size of the

[11] If the supports of the random variables s and d are finite, then the fed funds rate always equals r_{IOR} for a sufficiently large (abundant) reserve supply, and the central bank has perfect interest rate control. If the supports of the demand and supply shocks are not finite, e.g., when s and d are normally distributed, then the probability that the fed funds rate does not equal r_{IOR} can be made arbitrarily small by choosing a sufficiently large reserve supply.

central bank's balance sheet, then having (almost) perfect control over the policy rate may not constitute an optimal implementation framework, as we shall see in Section 3.5.

Before describing and solving the central bank's policy implementation problem, we first characterize the individual bank's reserve holdings problem.

3.4 Reserve Demand

In this section, we first provide the intuition underlying an individual bank's demand for reserves. We then consider a banking system with many banks and sum their individual demands to derive the banking system's demand for reserves (see, e.g., Poole (1968), Ennis and Keister (2008) and references therein). Our formal derivations are relegated to Appendix A.

The bank's problem. Bank i's problem can be characterized as follows: At $t = 5$, bank i chooses between borrowing and lending in the interbank market given its current desired reserves, $\bar{R}_i + d_i$, and knowing that after the interbank market closes at the end of the period, it will receive a late shock u_i. Since the value of the late shock is not known when the interbank market is open, bank i's demand for reserves in the fed funds market depends on the distribution of the late shocks (and not on the realization of the shock). As in Poole (1968), the shock may push bank i's reserves below its desired level.[12] In this case, bank i borrows from the central bank at a penalty rate, r_P, to bring its reserve holdings up to the desired level. If reserves exceed its desired level, then the bank does not borrow from the central bank. In either case, bank i deposits all of its reserves at the central bank and earns an interest on these balances equal to r_{IOR}.

Let us consider bank i's decision between lending and not lending an additional unit of reserves in the interbank market at time $t = 5$. If bank i lends, then the (gross) payoff associated with this trade is r, the interbank rate. In time period $t = 6$, bank i receives a shock u_i to its reserve holdings, R_i. If its new level of reserves $R_i + u_i$ falls short of its desired level, $\bar{R}_i + d_i$, i.e., if $R_i + u_i \leq \bar{R}_i + d_i$, then bank i must borrow reserves from the central bank

[12]Some recent models use alternative modeling approaches, such as search and bargaining or preferred habitat, in contrast to the perfect competition assumption in Poole (1968). See for example Afonso and Lagos (2015), Afonso et al. (2019), Armenter and Lester (2017), Schulhofer-Wohl and Clouse (2018), and Chen, Clouse, Ihrig, and Klee (2016). Still, demand for reserves typically originates from a required quantity of reserves and timing of shocks that are broadly consistent with Poole (1968).

at the rate r_P to make up for the shortfall. Hence, with probability $\Pr(R_i+u_i \leq \bar{R}_i+d_i)$ bank i's net payoff from lending a unit in the interbank market is the sum of (i) the interbank rate r from the unit of reserve it lent in the interbank market, (ii) the penalty rate, r_P, it pays from borrowing this unit back from the central bank, and (iii) the interest on reserves, r_{IOR}, it receives when the borrowed unit is deposited at the central bank. And with probability $\Pr(R_i+u_i > \bar{R}_i+d_i)$, the date $t = 6$ late shock does not push bank i's level of reserves below it's desired level and bank i net payoff is simply the interbank rate r it received on the unit it lent out. Thus, the expected net payoff from lending is

$$\Pr(R_i + u_i \leq \bar{R}_i + d_i)(r - r_P + r_{IOR}) + \Pr(R_i + u_i > \bar{R}_i + d_i)r. \tag{1}$$

Bank i is indifferent between lending and not lending an additional unit of reserves in the interbank market when the expected return to lending the additional unit (equation (1)) equals the expected return of not lending, which equals r_{IOR}. Rearranging this indifference condition, the competitive interbank market rate, r, can be expressed as a weighted average between the penalty rate, r_P, and the rate on reserves balances, r_{IOR}, where the weights capture the probability that bank i's reserves fall, or not, below its minimum level, i.e.,

$$r = r_P \Pr(R_i + u_i \leq \bar{R}_i + d_i) + r_{IOR} \Pr(R_i + u_i > \bar{R}_i + d_i). \tag{2}$$

For simplicity, we assume that u_i is uniformly distributed over the interval $[-U_i, U_i]$ for all i. This assumption implies that the size of the late shock can take any value in the interval $[-U_i, U_i]$ with the same probability and is zero outside this interval. It also implies that the probability that the late shock, u_i, is less than some value $z \in [-U_i, U_i]$ is given by $\Pr(u_i \leq z) = (z + U_i)/2U_i$; is less than some value $z < -U_i$ is $\Pr(u_i \leq z) = 0$; and is less than some value $z > U_i$ is $\Pr(u_i \leq z) = 1$. We can then express the probability that the size of the late shock is such that reserves, $R_i + u_i$, fall short of bank i's desired level of reserves, $\bar{R}_i + d_i$—and hence the bank borrows from the central bank—as

$$\Pr(R_i + u_i \leq \bar{R}_i + d_i) = \begin{cases} 1 & \text{if } R_i < \bar{R}_i + d_i - U_i \\ \dfrac{\bar{R}_i + d_i + U_i - R_i}{2U_i} & \text{if } \bar{R}_i + d_i - U_i \leq R_i \leq \bar{R}_i + d_i + U_i \\ 0 & \text{if } R_i > \bar{R}_i + d_i + U_i \end{cases} \tag{3}$$

Substituting equation (3) in equation (2) and rearranging terms, the demand for reserves of bank i when the interbank market opens at time $t = 5$ is given by

$$R_i = \begin{cases} [0, \bar{R}_i + d_i - U_i] & \text{if } r = r_P \\ \bar{R}_i + d_i + U_i - 2U_i \ \dfrac{r - r_{IOR}}{r_P - r_{IOR}} & \text{if } r_{IOR} < r < r_P \\ [\bar{R}_i + d_i + U_i, \infty) & \text{if } r = r_{IOR} \end{cases} \tag{4}$$

When the interbank rate equals the penalty rate, bank i is indifferent between holding any amount of reserves between zero and $\bar{R}_i + d_i - U_i$, because in this region of the demand curve bank i's reserve balances will always be below its desired level for any realization of the late shock. This creates a flat demand curve at $r = r_P$. When the interbank rate falls below the penalty rate ($r < r_P$), bank i demands more reserves in the interbank market, since borrowing from the central bank at rate r_P is more costly than in the interbank market at rate r. This generates a downward slopping demand curve until the interbank rate equals the interest on reserves. When $r = r_{IOR}$, the opportunity cost of demanding reserves in the interbank market is zero and bank i becomes indifferent between holding any amount of reserves greater than $\bar{R}_i + d_i + U_i$; in this region of the demand curve, bank i's reserve balances will always exceed its desired level for any realization of the late shock. At rate $r = r_{IOR}$, the demand curve for reserves becomes flat again.

Banking system demand for reserves. We can derive the banking system demand for reserves by simply summing the individual demand curves of the banks in the banking system, i.e., by summing the reserves demanded by the each bank for each interest rate. The aggregate demand for reserves, R^D, is given by

$$R^D = \begin{cases} [0, \bar{R} + d - U] & \text{if } r = r_P \\ \bar{R} + d + U - 2U \ \dfrac{r - r_{IOR}}{r_P - r_{IOR}} & \text{if } r_{IOR} < r < r_P \\ [\bar{R} + d + U, \infty) & \text{if } r = r_{IOR} \end{cases} \tag{5}$$

where $R^D \equiv \sum_i R_i$ is the aggregate quantity of reserves demanded by banks when the interbank market opens at $t = 5$; $\bar{R} = \sum_i \bar{R}_i$ is the initial aggregate quantity of of desired reserves (before the demand shocks, d_i, are realized); $d \equiv \sum_i d_i$ is the aggregate demand shock to the banking system's initial desired reserves (which is realized at $t = 4$, before the interbank market opens); and $U \equiv \sum_i U_i$ is sum of each bank i's maximum late period shock (the shock which is realized at $t = 6$, after the interbank market closes).

Finally, for simplicity, we do not consider the flat portion of the demand curve for very low levels of reserves, i.e. when $R^D \leq \bar{R} + d - U$, and treat the demand curve for low levels of reserves as a single downward sloping line.[13]

3.5 Reserve Supply

The equilibrium interest rate. The central bank understands the behavior of banks and the nature of the shocks s and d that hit the supply and demand for reserves of the banking sector. The time $t = 5$ equilibrium is characterized by the aggregate supply of reserves equating the aggregate demand for reserves, $R^D = R^S$, or, equivalently, $R^D = R + s + x$. Since R, s, x and d are known at $t = 5$, the equilibrium interbank interest rate is obtained by substituting $R + s + x$ for R^D in the aggregate demand function (5). Rearranging terms, the market-clearing interbank rate can be expressed as an implicit function of $R + s - d - x$, $r(R + s - d + x)$:

$$r = \begin{cases} r_P & \text{if } R + s - d + x < \bar{R} - U \\ r_{IOR} - c_0(R + s - d + x - \bar{R} - U) & \text{if } \bar{R} - U \leq R + s - d + x \leq \bar{R} + U \\ r_{IOR} & \text{if } R + s - d + x > \bar{R}_a + U \end{cases} \tag{6}$$

where $c_0 = (r_p - r_{IOR})/2U$. If $R + s - d + x \in (\bar{R} - U, \bar{R} + U)$, then the interbank rate is decreasing in the quantity of reserves in the banking system R. It is straightforward to show that the market clearing interbank rate has the expected comparative statics. In particular, $\partial r/\partial R \leq 0$, $\partial r/\partial r_p \geq 0$ and $\partial r/r_{IOR} \geq 0$.

The central bank's problem. When implementing monetary policy, the central bank incurs costs associated with: (i) the size of its time $t = 1$ balance sheet; (ii) undertaking open market operations at time $t = 3$; and (iii) "missing" its time $t = 5$ target interbank (fed funds) rate, i.e., the equilibrium interbank rate differs from the cental bank's target.[14] We assume that these costs are linear and that the central bank's monetary policy implementation

[13] Relaxing this assumption requires taking into account the flat segment of the demand at $r = r_P$, which captures an environment where (negative) shocks are very large, interbank rates equal the penalty rate, and banks borrow consistently from the central bank. This complicates the analysis considerably without providing any additional economic insight.

[14] Concerns about the size of central bank balance sheets are often raised in policy normalization discussions. For example, in 2014, the Federal Reserve expressed its intention to 'hold no more securities than necessary' in its "Policy Normalization Principles and Plans," available at `https://www.federalreserve.gov/monetarypolicy/policy-normalization-discussions-communications-history.htm`.

cost function is given by:

$$V \equiv E\left\{\alpha|r(R+s-d+x)-r(R)|+\beta|x|\right\}+\gamma R. \tag{7}$$

The cost function V is expressed from a time $t=1$ perspective where s, d, and x are random variables. The term $\alpha|r(R+s-d+x)-r(R)|$ represents the cost associated with interest rate volatility, i.e., the cost of missing the target rate, $r(R)$. Intuitively, if there are no shocks to neither the reserve supply nor the desired reserve holdings, the equilibrium interest would be $r(R)$ since $s=d=x=0$. Therefore, $r(R+s-d+x)-r(R)$ is the equilibrium deviation from the target interbank interest rate when the economy is characterized by uncertainty. The term $\beta|x|$ represents the operational cost of conducting open market operations. For simplicity, we assume that the cost is symmetric, i.e., draining and injecting reserves are equally costly. Finally, the term, γR, captures the political-economy cost associated with the size of the central bank's time $t=1$ balance sheet, where we assume that a higher level of reserves, R, results in higher political-economy costs. Without loss of generality, we set $\gamma=1$.

The central bank takes two policy implementation actions: at time $t=1$, it chooses the level of reserves R, and, then, at time $t=3$, it selects the size of the open market operation, $x(R,s)$, given its initial choice of reserves, R, and the realization of the time $t=2$ reserve supply shock s. Taking into account how it conducts open market operations for all possible combinations of (R,s), the central bank's choice of the $t=1$ reserve supply is given by the solution to

$$\min_{R\geq\bar{R}} E\{\alpha|r[R+s-d+x(R,s)]-r(R)|+\beta|x(R,s)|\}+R. \tag{8}$$

The restriction that $R\geq\bar{R}$ guarantees that the central bank supplies enough reserves to at least meet the initial desired reserves demanded by the banking system.[15] Although we treat $x(R,s)$ as an exogenous function, this function is constructed from the state-by-state time $t=3$ optimization problem.[16]

[15]From a technical perspective, we impose this constraint to rule out a solution where the central chooses a very low level of reserves, i.e., $R\ll\bar{R}-U$ can be the solution to the minimization problem in (8). This solution would be characterized by a very low political cost, very low interest rate volatility and no need for operations. The interest rate will be almost always equal to the penalty rate, r_P, and there would be significant borrowing from the central bank at time $t=6$. The central bank's objective function does not prevent this outcome from occurring because it does not place a cost associated with lending reserves to banks at date $t=6$.

[16]Alternatively, we could have represented the central bank's problem as choosing both R and x at time

The central bank's decisions regarding initial reserves, R, and open market operations, x, become trivial if the central bank does not face "real" trade-offs among the various costs. For example, if $\beta > \alpha c_0$, then the cost of an open market operation always exceeds the benefit associated with hitting the target rate, which implies that the central bank does not implement open market operations, i.e., $x = 0$. If, in addition, the cost of missing the target rate, α, is very small, then the central bank is not particularly concerned about interest rate volatility. In this case, in addition to never undertaking an open market operation, $x = 0$, the central bank always chooses the minimum possible level of reserves, $R = \bar{R}$. To rule out these pathological cases, we assume that $\beta < \alpha c_0$ and that β is "not too small."[17]

Parametrization of supply and demand shocks. We parameterize the shocks s and d in a stylized and convenient way to capture the uncertainty a central bank faces when it makes its reserve decisions at times $t = 1$ and $t = 3$. At time $t = 1$, the realizations of the supply and demand shocks are unknown. We model this uncertainty as the standard deviation of $s + d$, which we denote as σ. At time $t = 3$, after the supply shock has been revealed, we model the uncertainty that the central bank faces as the conditional standard deviation of $s + d$, and denote it by $\rho\sigma$. One can interpret ρ as how predictable $s + d$ is at time $t = 3$. Clearly, since the uncertainty about the supply shock has been resolved at this time, ρ captures the predictability of the demand shock.[18] For example, if $\rho = 0$, then the conditional standard deviation of the sum of the shocks is zero, and the demand shock, d, is perfectly predictable and equal to 0. For simplicity, we assume that s and d are independently distributed with

$$s \sim \mathcal{N}\left(0, (1-\rho^2)\sigma^2\right) \tag{9}$$

$$d \sim \mathcal{N}\left(0, \rho^2\sigma^2\right), \tag{10}$$

where $\sigma \in (0, \bar{\sigma})$. Note that the demand shock d is an aggregate of the individual demand shocks, $d = \sum_i d_i$. One interpretation is that $(d_1, d_2, ..., d_N)$ follows a multivariate normal

$t = 1$, where x is a function of s. This formulation, however, is not as convenient as the one we propose since it would not give rise to a unique functional form for $x(s)$ without additional restrictions. For example, one can make an arbitrary deviation over a measure zero set that does not change the expected value. Furthermore, given the smoothness of the problem at time $t = 3$, we do not need to worry about the measurability of x if we write $x(R, s)$ as an optimizer for the time $t = 3$ problem.

[17] We formally define "not to small" in Appendix B. In some proofs, we impose a slightly stronger version of the inequality $\beta < \alpha c_0$; see Appendix B.

[18] Alternatively, we can parameterize the shocks using the standard deviations, say σ_s and σ_d, of s and d, respectively. The advantage of the current setup is that it clearly incorporates the concept of predictability, as discussed in Section 2.1, and allows us to describe the outcome of changes in predictability.

distribution, which implies that d follows a normal distribution. For simplicity, we also assume that uncertainty about the demand and supply shocks, σ, is not too large compared to the aggregate late shock U.[19]

We now turn to the central bank's choice of monetary policy implementation regime.

3.6 Policy Implementation without Demand Uncertainty

We first consider the case with no uncertainty about the demand for reserves and assume that demand shocks are perfectly predictable, i.e., $\rho = 0$. In practice, this scenario describes the U.S. reserve market in the pre-crisis period. In that period, banks demanded reserves mainly to satisfy reserves requirements and the Trading Desk at the New York Fed was able to accurately estimate the banking system demand for reserves.

Perfect predictability of demand implies that all uncertainty is resolved by time the central bank conducts open market operations at time $t = 3$. Given the initial supply of reserves, R, and the realization of the supply shock, s (and $d = 0$), the central bank chooses the size of its open market operation, x, to minimize the cost of implementing monetary policy, $V(R)$, in (7), i.e.,

$$x(R, s) = \arg\min_{R \geq \bar{R}} \left\{ E[\alpha | r(R + s + x) - r(R)|] + \beta |x| + R \right\},$$

where we impose $R \geq \bar{R}$ to ensure that the central bank supplies enough reserves to meet the desired amount demanded by the banking system. We also impose $\beta < \alpha c_0$ to guarantee that the central bank is willing to undertake open market operations and assume that $\beta > \bar{\beta}$—which sets a floor on the cost of open market operations—so that the central bank does not simply choose the minimum level of initial reserves and then adjusts reserves through injections because open market operations are cheaper than supplying reserves at time $t = 1$.[20] In addition, we assume a lower bound on the ratio $U/\bar{\sigma}$.[21] Proposition 1 summarizes

[19]Otherwise we would not be able to abstract away from the flat portion of the demand curve at $r = r_P$, where banks believe that they will be borrowing reserves from the central bank with certainty. Also, if σ were too large, it would be difficult to define the concept of scarce reserves because even at the minimal level of reserve supply, $R = \bar{R}$, there would be a substantial chance that r might equal r_{IOR}. Assumption A3 in Appendix B formally defines "not too large."

[20]See Appendix B, assumption A1. In the case of no demand uncertainty, $\bar{\beta} \geq 2$ is sufficient for deriving the results.

[21]This assumption—A3 in Appendix B—is necessary for the concept of scarce reserves to be well-defined. See footnote 19.

our first result.

Proposition 1. *Assume $\rho = 0$. Then, the cost of implementing monetary policy, $V(R)$, has two local minima: one at $R = \bar{R}$ and another, denoted R_A, at $R > \bar{R} + U$.*

The proofs to all propositions are in Appendix B. Notice that $R_A > \bar{R} + U$ means that the higher value local minimum is located beyond the "kink" of the aggregate demand curve for reserves in equation (6).

Next, we describe the choice between $R = \bar{R}$ and $R = R_A$ first using an illustrative example based on our model, and then more generally based on theoretical implications of our model.

Scarce, ample and abundant reserves. In the following example, we define reserves as being scarce, ample, or abundant depending on how likely it is that the supply shock s pushes reserves below the threshold $\bar{R} + U$ that defines the transition from the flat to the downward-sloping region of the demand curve (see equation 6). When the supply of reserves exceeds this threshold the equilibrium interbank interest rate equals the rate that the central bank pays on reserves, r_{IOR}; when the supply of reserves falls short of this threshold, the equilibrium interbank interest rate exceeds r_{IOR}. We define reserves as *abundant* when there is a "negligible" probability that the time $t = 1$ level of reserves, R, falls below $\bar{R} + U$ after the realization of the supply shock s; *ample* when the probability is "reasonably small"; and *scarce* when the probability is greater than reasonably small.

In the calibration exercise, we define a negligible probability to be less than or equal to 0.1 percent; a reasonably small probability to be between 0.1 and 15 percent; and greater than reasonably small probability as exceeding 15 percent. Figure 3 provides an illustration of Proposition 1. In the example described by the figure, the lower limit for an abundant reserve supply is equal to $\bar{R} + U + 3.1 \times \sigma$, and the lower limit for ample reserve supply is equal to $\bar{R} + U + 1.0 \times \sigma$. Hence, an abundant reserve supply is at least $2.9 \times \sigma$ larger than the lowest level of ample reserves in this example.

The central bank chooses between the two level of reserves—scarce or ample—in Proposition 1 by comparing the expected cost of implementing monetary policy with scarce reserves, $R = \bar{R}$, and with ample reserves, $R = R_A > \bar{R} + U$. In our calibration, the central bank chooses an ample level of reserves at time $t = 1$ (see Figure 3), since its expected cost is

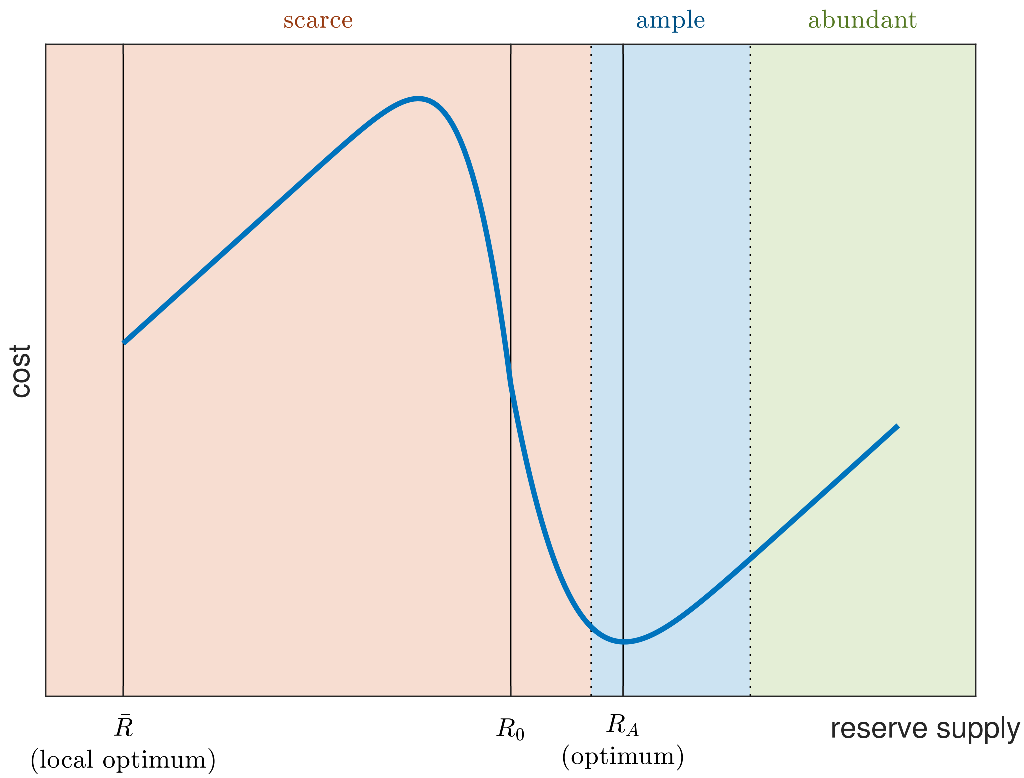

Figure 3: Central Bank's Cost with No Demand Shocks ($\rho = 0$)

The blue line shows the central bank's expected cost as a function of the initial reserve supply (equation (7)). The cost is generated under the assumption that there are no demand shocks, $\rho = 0$. The solid vertical lines mark the two local optima of the cost function—scarce ($\bar{R}$) and ample (R_A)—and the location of the kink (R_0) in the demand curve; the kink's location is deterministic because there are no demand shocks. The dotted lines mark the boundaries between scarce (red-shaded), ample (blue-shaded), and abundant (green-shaded) reserve supply based on the example discussed in Section 3.6.

less than that associated with implementing monetary policy with scarce reserves $R = \bar{R}$. In general, however, the central bank's choice need not be $R = R_A$ and instead might be $R = \bar{R}$. Next, we discuss what determines this choice in our model.

Optimal level of reserves. Before discussing the optimal level of reserves, we first characterize the local minimum at R_A and then turn to the central bank's choice of reserves. We can determine the level of reserves R_A by equating the marginal benefit of choosing a higher level of initial reserves—which would be associated with smaller open market operations—to its marginal cost—which is the political-economy cost of a large balance sheet. After the supply shock, s, is realized, the central bank conducts an open market operation to inject reserves only if the new level of reserves, $R+s$, is less than the threshold $\bar{R}+U$. The size of the operation will be $x=-s-[R-(\bar{R}+U)]$ which equates the equilibrium interbank interest rate to the interest on reserve balances, i.e., $r(R+s+x)=r_{IOR}$. Clearly, increasing the time $t=1$ supply of reserves, R, reduces the size of open market operations at $t=5$ only if $R+s<\bar{R}+U$; when $R+s>\bar{R}+U$ the central bank does not conduct any open market operations ($x=0$). This implies that the marginal benefit of increasing R is equal to $\beta \Pr[R+s<\bar{R}+U]=\beta\Phi[(\bar{R}+U-R)/\sigma]$. Since the marginal cost of increasing reserves is equal to $\gamma=1$, the level of reserves that minimizes the cost of implementing monetary policy is obtained by equating the marginal benefit to the cost of increasing reserves at date 1, which yields the following expression for R_A:[22]

$$R_A=\bar{R}+U+\sigma\Phi^{-1}(1-\frac{1}{\beta}). \tag{11}$$

Intuitively, R_A is increasing in β because a higher cost of open market operations incentivizes the central bank to increase the initial level of reserves R so as to move farther away from the threshold, $\bar{R}+U$. This threshold determines the transition between the negatively sloped and flat regions of the demand curve. R_A is also increasing in the volatility of the demand and supply shocks σ: As σ increases so does the probability that reserves will be pushed below the threshold $\bar{R}+U$, incentivizing the central bank to increase R.[23]

We now compare the central bank's cost of implementing a scarce reserve framework with $R=\bar{R}$ to the cost of implementing an ample reserve framework with $R_A>\bar{R}+U$ given by equation (11):

- **Scarce reserves.** The central bank's implementation cost evaluated at $R=\bar{R}$ is approximately equal to

$$V(R^*=\bar{R})\approx\sqrt{\frac{2}{\pi}}\beta\sigma+\bar{R}. \tag{12}$$

  The first term captures the cost of fully offsetting reserve supply shocks, i.e., it is the expected value of $\beta|-s|$.[24]

[22]As discussed in Section 3.4, we do not consider the region of the demand curve with very low reserve balances, i.e., the region to the left of $\bar{R}+d-U$. The missing term is very small and does not provide additional insights.

[23]Note that we assume $\beta>2$ (see assumption A1 in Appendix B); otherwise, the cost of open market operations will be too low and the central bank will always choose the lowest level of reserve supply; as a result, there will be no local minimum at $R>\bar{R}+U$, and R_A will not be defined.

[24]This result is an approximation. In an extreme event of a very large shock, the central bank will choose

- **Ample reserves.** The central bank's implementation cost evaluated at $R = R_A$ is

$$V(R^* = R_A) = \phi\left(\Phi^{-1}\left(1 - \frac{1}{\beta}\right)\right)\beta\sigma + \bar{R} + U. \tag{13}$$

  As above, the first term is the expected cost associated with open market operations.

The first term in equation (13) is smaller than the first term of (12) because the central bank's open market operations are smaller and less frequent when reserves are ample. Since both of these terms are linear in σ—and $R_A > \bar{R}$, if σ is "large enough," the ample-reserve regime will have a lower expected implementation cost and, hence, will be preferred to the scarce-reserve regime. Proposition 2 formalizes this intuition:

Proposition 2. *Assume $\rho = 0$ and $\sigma \in (0, \bar{\sigma}]$. There exists a critical value $\sigma^* \in (0, \bar{\sigma}]$ such that if $\sigma < \sigma^*$, then $R = \bar{R}$ is the cost minimizing reserve level and if $\sigma > \sigma^*$, then $R = R_A$ is the cost minimizing level.*

Note that the constraint $\sigma \leq \bar{\sigma}$ ensures that σ is not too large, as discussed earlier (see assumption A3 in Appendix B or footnote 19).

There is no closed-form solution for σ^*, but we can derive an approximate expression by finding the value of σ that equates the approximation of $V(\bar{R})$ in equation (12) to $V(R_A)$ in equation (13):

$$\sigma^* \approx [\sqrt{\frac{2}{\pi}} - \phi\left(\Phi^{-1}\left(1 - \frac{1}{\beta}\right)\right)]^{-1}\frac{U}{\beta}. \tag{14}$$

When demand shocks are predictable, $\rho = 0$, and the volatility of the shocks to the reserve supply is "low" ($\sigma < \sigma^*$), then a scarce-reserve regime will be optimal. This scenario is consistent with the monetary policy implementation regime that prevailed in the U.S. prior to the 2007-2009 financial crisis. On the other hand, when the volatility of the supply shock is "high," then the central bank will choose a regime with ample reserves. This scenario is consistent with monetary implementation policy in the U.S. since the 2007-2009 financial crisis. Figure 4 illustrates this intuition: When the volatility σ is low—the orange line—it is less costly for the central bank to implement monetary policy with scarce reserves, $R = \bar{R}$; when volatility is high—blue line—ample reserves, $R \equiv R_A > \bar{R} + U$, provides the implementation framework with the lowest cost.

not to offset it. However, the probability of this event is very small because we assume that U is large relative to σ. Appendix B derives the exact formulation; the economic intuition and mathematical arguments remain the same.

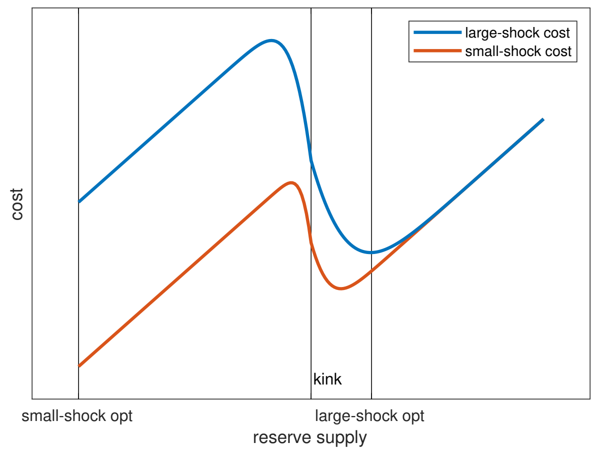

Figure 4: Central Bank's Cost for Different Values of Shocks (σ)

The blue and orange lines show the central bank's expected cost as a function of the initial reserve supply (equation (7)) for two values of the standard deviation of the (sum of) demand and supply shocks σ. The blue (orange) line corresponds to a high (low) value of σ. For high σ, the central bank chooses an implementation with ample reserves ('large-shock opt') while for low σ, the implementation cost is lower with scarce reserves ('small-shock opt'). Costs are generated under the assumption that there are no demand shocks, $\rho = 0$.

Intuitively, σ^* is higher if U is higher. If the threshold (kink) in the reserve demand is farther away from the minimum level, then the central bank has a stronger incentive to choose the minimum level of reserve supply to avoid incurring the cost associated with a larger reserve supply, which is proportional to U, while the benefit of smaller open market operations does not depend on U.

The cost of engaging in open market operations, β, is also an important determinant of the optimal monetary policy implementation framework that the central bank chooses. Intuitively, if the cost of conducting open market operations, β, is relatively small, then the central bank will choose a framework with scarce reserves even at higher levels of volatility, σ, using open market to adjust reserves if needed. Alternatively, if the operational costs β are relatively high, then abundant reserves will be chosen for even lower levels of volatility σ. This intuition is verified in the following proposition,

Proposition 3. *Assume $\rho = 0$. Then, $\partial\sigma^*/\partial\beta \leq 0$. This inequality is strict, $\partial\sigma^*/\partial\beta < 0$, if $\sigma^* < \bar{\sigma}$.*

Since the 2007-2009 financial crisis, reserve supply shocks and the required size of open market operations to offset those shocks have become substantially larger (see section 2.2). Even though the cost of open market operations is proportional to its size in the model, in practice this cost might be convex. For example, with larger operations, the central bank might be much more concerned about risks associated with counterparty exposures, potential market distortions, possible mistakes, and so on. If so, the post-crisis environment could be described with a higher cost parameter β, which makes an ample-reserve regime more preferable to the central bank.

3.7 Environment with Demand Uncertainty

We now consider the case where there is uncertainty about the demand for reserves—i.e., $\rho \neq 0$—at the time $t = 5$ when the central bank conducts an open market operation. We can interpret this scenario as representative of the U.S. reserve market in the post-crisis period. As discussed in Section 2.2, since the 2007-2009 financial crisis, changes in bank regulation, banks' risk management practices and liquidity stress tests, among other factors, have transformed the banking system's demand for reserves. We interpret the post-crisis period as characterized by an increase in the magnitude of the demand and supply shocks—a larger σ—and a decline in the predictability of the reserve demand—a larger ρ.

An increase in uncertainty and a decrease in predictability may lead a central bank to prefer an ample reserve supply to a scarce one. When reserves are scarce and the reserve demand becomes less predictable (ρ increases), open market operations become less effective in stabilizing interest rate movements, making an ample-reserve regime more attractive to the central bank. The following simple example illustrates this point. Suppose first that the central bank chooses an initial level of reserves that is scarce and equal to 3, i.e $R = \bar{R} = 3$ at $t = 1$, and that a supply shock s of -1 has been realized at $t = 2$, where the supply shock can be either -1 or 1 with equal probability. Since the reserve demand curve in the scarce region is downward sloping, in the absence of reserve demand shocks, the central bank would inject $x = 1$ unit of reserves at $t = 3$. This operation costs the central bank β but saves αc_0 since the equilibrium interbank rate equals the target rate $r(R + s + x = 3)$.

Now suppose that, at $t = 4$, there are demand shocks d of either -1 or $+1$ with equal probability and that the central bank injects a unit of reserves at time $t = 3$. Then the equilibrium interest rate will be either $r(2) > r(3)$ if $d = 1$ or $r(4) < r(3)$ if $d = -1$. The time $t = 3$ expected cost associated with $x = 1$ is $\beta + \alpha c_0$. If, instead, the central does not inject a unit of reserve at time $t = 3$, then the equilibrium interest rate will be either $r(1) > r(3)$ if $d = 1$ or $r(3)$ if $d = -1$. The expected cost associated with $x = 0$ is αc_0. Hence, the central bank's best response at time $t = 3$ is *not* to undertake an open market operation, i.e., $x = 0$ since the expected cost to the central bank is lower by β relative to the cost of conducting the operation ($x = 1$).

We consider the case where the central bank chooses ample reserves equal to $R_A = \bar{R}+U+2$ at $t = 1$ instead of scarce reserves for this example. Independently of the supply and demand shock realizations, the $t = 5$ equilibrium interbank rate is $r(R_A + s + x - d) = r_{IOR}$: hence, there is no interest rate variability. In this example, the central bank will choose ample reserves if the increase in balance sheet costs, $R_A - \bar{R}$, is less than the cost savings associated with interest rate variability, αc_0.

To see that the decrease in the predictability of demand makes ample reserves more attractive, we assume that the central bank learns what the demand shock at $t = 5$ will be at $t = 3$. Because the demand shock is known at $t = 3$, the central bank will offset the demand shock at $t = 3$ when reserves are scarce. The cost savings from choosing ample reserves at date $t = 1$ is only equal to β because the reserve injection at $t = 3$ would be 2 or 0 with equal probability, which is smaller than αc_0. Thus, cost savings from choosing ample reserves are larger with less predictable demand.

Qualitatively, key results and insights from the model with predictable reserve demand hold when the reserve demand is no longer perfectly predictable. Importantly, the central bank's $t = 1$ reserve supply decision is still a choice between a scarce-reserve regime and an ample-reserve regime. In particular,

Proposition 4. *The cost function $V(R)$ has two local minima. One is at $R = \bar{R}$ and another, denoted R_A, is at $R > \bar{R} + U$.*

The central bank's cost function retains the same basic shape as in the model with predictable demand (Figure 3).

The cost that a central bank incurs from interest rate volatility is mitigated in an ample reserve regime since the equilibrium interbank interest rates are less responsive to demand shocks when reserves are ample relative to environments with scarce reserves. This suggests that for a given σ, a central bank would favor a scarce (ample) reserve regime, the higher (lower) is the predictability of shocks. And, for a given level of predictability, a central bank would choose a scarce (ample) reserve regime the lower (higher) is the value of σ. The following proposition verifies that this intuition is correct.

Proposition 5. *For $\sigma \in (0, \bar{\sigma}]$ and $0 \leq \rho \leq 1$, there exists a function $F(\rho) \in (0, \bar{\sigma}]$ such that if $\sigma < F(\rho)$, then $R = \bar{R}$ is optimal, and if $\sigma > F(\rho)$, then $R = R_A$ is optimal. F is monotonically decreasing in ρ.*

An important takeaway from this proposition is that an increase in either σ or ρ makes an ample reserve supply more favorable (relative to a scarce supply). Figure 5 illustrates this result. When uncertainty is low and predictability is high—the orange line—the scarce reserve regime is preferred; when predictability is reduced (for the same level of uncertainty)—the blue line—the ample reserve regime is preferred.

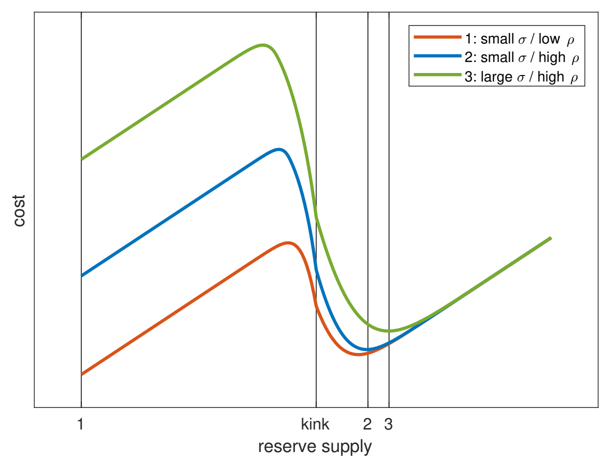

Figure 5: Central Bank's Cost: Cross-Model Comparison

The orange, blue, and green lines show the central bank's expected cost as a function of the initial reserve supply (equation (7)) for different values of σ and ρ. The orange line corresponds to a small value of σ and a low ρ (model 1: low uncertainty and high predictability); the blue line to a higher ρ than model 1 (model 2: low uncertainty and low predictability); and the green line to a larger σ than model 2 (model 3: high uncertainty and low predictability). The optimal supply of reserves is scarce for model 1 and ample for models 2 and 3. The vertical lines marked by numbers 1, 2 and 3 show the optimal reserve supply for models 1, 2 and 3, respectively. The expected location of the kink, $R = \bar{R} + U$, is also shown.

Figure 5 also shows that higher uncertainty and less predictability (the green line), consistent with the post-crisis period, would incentivize the central bank to move to an ample reserve regime from a scarce reserve regime such as the pre-crisis period (when uncertainty was low and predictability high). The optimal ample reserve supply in this case is greater than the supply associated with the blue line due to the higher uncertainty.

4 An Ample Reserve Regime in Practice

Our theory provides a rationale for the FOMC's decision in 2019 to remain in an ample reserve regime. In this section, we briefly discuss how well this regime has been at maintaining interest rate control, which is the primary objective of a monetary policy implementation framework. We also discuss some other benefits of this type of framework.

4.1 Interest Rate Control

Measuring the effectiveness of monetary policy implementation can be approached in many ways. One approach is to track the position of the policy rate relative to the target rate or range, and determine how often the rate deviates from the target. In the U.S., the effective federal funds rate has printed outside the target range in only two instances since the FOMC announced the establishment of a target range for the federal funds rate in 2008.[25]

Other measures of effectiveness focus on rate dispersion. Duffie and Krishnamurthy (2016), for instance, propose an index intended to capture rate dispersion across different segments of money markets.[26] Building on the Duffie-Krishnamurthy index, Afonso et al. (2017) show that the implementation framework used by the Federal Reserve in the post-crisis period has been effective and achieved good pass-through.

Another way of measuring the effectiveness of the operating regime is by the pass-through of administered rates to market rates. A particularly interesting approach to look at the effectiveness of the Federal Reserve's current implementation framework is to consider the

[25]The effective fed funds rate printed below the target range on December 31, 2015 and above the target range on September 17, 2019.

[26]In particular, Duffie and Krishnamurthy (2016) consider the volume-weighted average absolute deviation from the volume-weighted average rate, which captures how much each market rate deviates from the average rate across markets. To implement the index, the authors adjust rates for term and credit spreads, and weight each instrument's influence by its outstanding amount.

effect of the "technical adjustments" on various money market rates. A technical adjustment is a change to the administered rates—the interest on reserve balances (IORB) rate and/or the ON RRP rate—that is intended to foster trading in the fed funds market at rates well within the target range, rather than change the stance of monetary policy. Afonso et al. (2022) show that changes in administered rates, through technical adjustments, pass-through fully to other short-term rates.

Overall, the Federal Reserve's current implementation framework has been effective at interest rate control and at ensuring pass-through of the policy rate to short-term money markets rates.

4.2 Some Financial Stability Considerations

In this section, we discuss some financial stability implications of implementation frameworks with a large supply of reserve balances.

A monetary policy implementation regime with a sufficiently large supply of reserves allows banks to meet some of their needs for high-quality liquid assets (HQLA) with reserves. It also provides enough reserves for banks to meet their outflow needs with reserves during a time of stress and avoid the potential fire-sale effects from monetizing large quantities of assets (Bush et al., 2019). This makes the financial system safer, more resilient, and may reduce the need for banks to borrow from the central bank.[27]

Another benefit of a regime with a large supply of reserves is that reserves as "money-like" short-term safe assets are particularly attractive to some investors and, for that reason, carry a premium that reduces their yield. When the supply of money-like assets is too small, private sector participants have an incentive to issue liabilities that have money-like properties because of their low cost. This can result in excessive maturity transformation, which makes the financial system more fragile (Greenwood et al., 2016; Carlson et al., 2016).

Acharya et al. (2022), however, point to potential fragility concerns associated with cen-

[27] An alternative approach to avoid fire-sales effects would be for the central bank to offer a liquidity facility to "monetize" Treasury securities and, perhaps, other HQLA. See Andolfatto and Ihrig (2019), Liang and Parkinson (2020), and Group of Thirty Working Group on Treasury Market Liquidity (2021), for example. Such an alternative would be effective only if the liquidity facility does not suffer from stigma. The FOMC announced in July 2021 the establishment of a standing repo facility (see https://www.federalreserve.gov/newsevents/pressreleases/monetary20210728b.htm.)

tral bank balance sheet expansions. During expansions of central bank balance sheets, which ultimately create reserve balances, bank demandable deposits and lines of credit increase. This increase is not reversed during balance sheet shrinkage. The authors argue that this asymmetry is responsible for tightening liquidity conditions and stress episodes during balance sheet normalization and can make the banking system more dependent on central bank liquidity infusions during stress.

5 Conclusion

The 2007-2009 financial crisis, and its aftermath, have led to profound changes in the way many central banks implement monetary policy. In particular, large-scale asset purchases resulted in high levels of reserves balances at major central banks. Traditional implementation tools became ineffective, and central banks transitioned to control interest rates with administered rates, using a "floor" system. Some central banks, such as the Federal Reserve, have indicated that they expect to continue using this type of implementation framework in the foreseeable future.

The results of our paper shed some light on the implications of these policy implementation decisions. In this paper, we proposed a model of the banking system demand for reserves to study the trade-offs that policy makers face when choosing a monetary policy implementation framework. We reviewed key features of a central bank operating regime and discussed the costs and benefits of different implementation frameworks. We highlighted potential trade-offs between the size of a central bank's balance sheet and effectiveness of rate control, as well as the size of central bank operations. We showed that in the post-financial crisis environment, the optimal monetary policy regime is one where reserve balances lie between scarce and abundant.

Appendix

A Demand for Reserves

In this section we formally describe individual banks' problem and explain in more detail some of the results in Section 3.4. Bank i maximizes the following objective function:

$$\int_{-\infty}^{+\infty} V_i(u)\mu_i(u)du - R_i r. \tag{A-1}$$

μ_i is the probability density function of u_i and $V_i(u)$ is the value associated with the outcome $u_i = u$. $V_i(u)$ can be derived by integrating the marginal value of reserves up to $R_i + u_i$. The marginal value of reserves is r_P for $R_i + u_i < \bar{R}(i) + d_i$ and r_{IOR} for $R_i + u_i \geq \bar{R}(i) + d_i$. Integrating the marginal value from 0 (or any constant) to $R_i + u_i$, the expression for $V_i(u)$ is

$$V_i(u) = min(R_i + u_i, \bar{R}(i) + d_i)r_P + max(R_i + u_i - \bar{R}(i) + d_i, 0)r_{IOR}. \tag{A-2}$$

Note that

$$\begin{aligned} \frac{\partial V_i(u)}{\partial R_i} &= r_P \text{ if } R_i + u_i < \bar{R}(i) + d_i; \\ &= r_{IOR} \text{ otherwise.} \end{aligned} \tag{A-3}$$

The first-order condition (FOC) for bank i with respect to R_i is

$$\begin{aligned} 0 = \frac{\partial}{\partial R_i}[\int_{-\infty}^{+\infty} V_i(u)\mu_i(u)du - R_i r] &= \int_{-\infty}^{+\infty} \frac{\partial V_i(u)}{\partial R_i}\mu_i(u)du - r \\ = Prob(R_i + u_i < \bar{R}(i) + d_i)r_P &+ Prob(R_i + u_i \geq \bar{R}(i) + d_i)r_{IOR} - r. \end{aligned} \tag{A-4}$$

This is bank i's FOC described in Section 3.4.

Given the FOC of bank i, we can derive its demand for reserves. Recall that u_i is uniformly distributed over $(-U_i, U_i)$.

- If $R_i \leq \bar{R}(i)+d_i-U_i$, then $Prob(R_i+u_i < \bar{R}(i)+d_i) = 1$. Therefore, $r = 1\cdot r_P+0\cdot r_{IOR} = r_P$.
- If $R_i \geq \bar{R}(i) + d_i + U_i$, then $Prob(R_i + u_i < \bar{R}(i) + d_i) = 0$. Therefore, $r = r_{IOR}$.

- If $\bar{R}(i) + d_i - U_i \leq R_i \leq \bar{R}(i) + d_i + U_i$, then $Prob(R_i + u_i < \bar{R}(i) + d_i) = -(R_i - \bar{R}(i) - d_i - U_i)/(2U_i)$. Therefore,

$$\begin{aligned} r &= r_P(-\frac{R_i - \bar{R}(i) - d_i - U_i}{2U_i}) + r_{IOR}(1 + \frac{R_i - \bar{R}(i) - d_i - U_i}{2U_i}) \\ &= r_{IOR} + (r_P - r_{IOR})(-\frac{R_i - \bar{R}(i) - d_i - U_i}{2U_i}). \end{aligned} \quad \text{(A-5)}$$

Solving this equation for R_i, we have $R_i = \bar{R}(i) + d_i + U_i - 2U_i(r - r_{IOR})/(r_P - r_{IOR})$.

Next, we aggregate reserve demand across banks; recall that $R_{agg} = \sum_i R_i$, $\bar{R} = \sum_i \bar{R}(i)$, $d = \sum_i d_i$ and $U = \sum_i U_i$.

- If $R_{agg} \leq \bar{R} + d - U$, then $r = r_P$ in equilibrium and $R_i \leq \bar{R}(i) + d_i - U_i$ for every bank; note that R_i is not uniquely determined. To prove this, suppose that $R_j > \bar{R}(j) + d_j - U_j$ for some j. Then, to satisfy bank j's FOC, $r < r_P$. This implies $R_i > \bar{R}(i) + d_i - U_i$ for all i, implying $R > \bar{R} + d - U$, which is a contradiction.

- If $R_{agg} \geq \bar{R} + d + U$, then $r = r_{IOR}$ in equilibrium and $R_i \geq \bar{R}(i) + d_i + U_i$ for every bank; note that R_i is not uniquely determined. This can be proved by contradiction, similarly to how the previous case was proved.

- If $\bar{R} + d - U \leq R_{agg} \leq \bar{R} + d + U$, then $r_{IOR} < r < r_P$, because otherwise, R_{agg} would be outside the range. For any r in (r_{IOR}, r_P), bank i's choice of R_i is unique and given by $R_i = \bar{R}(i) + d_i + U_i - 2U_i(r - r_{IOR})/(r_P - r_{IOR})$, as shown earlier. Summing this expression across i, we have

$$R_{agg} = \bar{R} + d + U - 2U(\frac{r - r_{IOR}}{r_P - r_{IOR}}). \quad \text{(A-6)}$$

Solving this for r, we have

$$r = r_{IOR} - (\frac{r_P - r_{IOR}}{2U})(R_{agg} - \bar{R} - d - U) \text{ if } \bar{R} + d - U < R_{agg} < \bar{R} + d + U. \quad \text{(A-7)}$$

In equilibrium, $R_{agg} = R + s + x$ and recall $y \equiv R + s + x - d$. Writing the expression in terms of y, we have

$$r = r_{IOR} - (\frac{r_P - r_{IOR}}{2U})(y - \bar{R} - U) \text{ if } \bar{R} - U < y < \bar{R} + U. \quad \text{(A-8)}$$

This is the functional form of $r(y)$ described in Section 3.4.

B Technical Assumptions and Proofs

This section has proofs associated with Sections 3.6 and 3.7. We first state three technical assumptions on model parameters:

A1. $\beta > 1/\Phi(-\sqrt{2log2})$: β represents the unit cost of conducting market operations; if it were very small, the central bank would always choose the minimal level of reserves, $R = \bar{R}$. Except for proving Proposition 5, $\beta > 2$ is sufficient; note that Φ is the cumulative distribution function of the standard normal distribution, and $1/\Phi(-\sqrt{2log2})$ is about 8.[28]

A2. $1.1 < \alpha c_0/\beta < 2$: $\beta < \alpha c_0$ means that the central bank would rather use operations to reduce interest rate volatility than let the rate move. $\alpha c_0 < 2\beta$ helps simplify the central bank's problem. Except for proving Proposition 5, a lower bound of 1 on $\alpha c_0/\beta$ is sufficient.

A3. $U/\bar{\sigma} > \Phi^{-1}(1 - \frac{2}{3\alpha c_0})$: As discussed in Section 3.5, this helps simplify the central bank's problem.

Proof of Proposition 1: The proposition is as follows: Assume that $\rho = 0$. The cost function $V(R)$ has two local minima: $R = \bar{R}$ (scarce reserves) and another in $R > R_0$ (ample reserves).

Technically proving this proposition is redundant because we will prove a more general Proposition 4. Nonetheless we still provide a proof because it is helpful in understanding the choice of the central bank in the equilibrium. Recall that the central bank's cost function from date 0 perspective is

$$V(R) = E[\alpha|r(R + s - d + x(R, s)) - r(R)| + \beta|x(R, s)|] + R. \tag{B-9}$$

The central bank seeks to minimize this cost function under the constraint $R \geq \bar{R}$, with $x(R, s)$ optimal in each state s. To show the existence of the two local minima, it is sufficient to show the following:

- $V'' < 0$ for $R < R_0$.

[28] In particular, if $\beta < 1$, the central bank always chooses $R = \bar{R}$ because state-contingent reserve injections are a cheaper way to supply reserves. This is an unrealistic assumption.

- $V'' > 0$ for $R > R_0$.

- $V'(\bar{R}) > 0$.

- $V'(R_0+) < 0$, where $V'(R_0+)$ denotes the right limit of V' at R_0.

- $\lim_{R\to\infty} V'(R) > 0$.

Note $R_0 = \bar{R} + U$. The third inequality shows that $R = \bar{R}$, the scarce supply, is a local optimum and the first inequality shows that there is no other optimum below the kink level, $R < R_0$. The fourth and the fifth inequality show that there is an optimum with $R > R_0$ (the ample supply), and it is unique by the second inequality.

To prove the first inequality, $V'' < 0$ for $R < R_0$, we recognize that increasing R is less costly for R closer to R_0 (while still $R < R_0$) because the flat portion of the demand curve helps lower the cost due to interest rate volatility. As stated in the main text, we use the following approximate form for $r(y)$:

$$\begin{aligned} r(y) &= r_{IOR} \text{ if } y > R_0; \\ &= r_{IOR} + c_0(R_0 - y) \text{ otherwise.} \end{aligned} \tag{B-10}$$

We first derive the optimal $x(R, s)$ for $R < R_0$. Note that the problem is

$$\min_x \alpha|r(R+s+x) - r(R)| + \beta|x|. \tag{B-11}$$

d is dropped because $\rho = 0$ and R is dropped from the cost function because this is a problem of determining x taking R and s is given. Assumption A2 implies $\beta < \alpha c_0 < 2\beta$ and given these inequalities, we can characterize x:

- If $s \leq R_0 - R$, then $x = -s$. $R + s$ is still in the steep portion of the demand curve and $\beta < \alpha c_0$ implies that the central bank finds it optimal to completely offset s.

- If $s \geq R_0 - R$, then two cases are possible. First, note that $\beta < \alpha c_0$ implies that either $x = -s$ or $x = 0$. If $R + s + x < R_0$ and $R + s + x \neq R$, then the central bank can further its reduce cost by setting $s + x = 0$ because $\beta < \alpha c_0$. If $R + s + x \geq R_0$ and $x \neq 0$, it can reduce cost by setting $x = 0$ simply because $\beta > 0$. Thus the optimal choice is the better one between $x = -s$ and $x = 0$. The central bank is indifferent between these two choices if

$$\alpha c_0(R_0 - R) = \beta s. \tag{B-12}$$

For $s < (R_0 - R)\alpha c_0/\beta$, the optimal choice is $x = -s$; for $s > (R_0 - R)\alpha c_0/\beta$, it is $x = 0$.

Recall that $V(R) = E[\alpha|r(R+s-d+x(R,s)) - r(R)| + \beta|x(R,s)|] + R$. The first term of this expression is

$$E\alpha|r(R+s-d+x(R,s)) - r(R)| = \alpha c_0(R_0 - R)\Phi(-\frac{1}{\sigma}\frac{\alpha c_0}{\beta}(R_0 - R)). \tag{B-13}$$

This is a result of applying the explicit formula for x that we just derived. Similarly, we can calculate the second term:

$$E\beta|x(R,s)| = \beta \int_{-\infty}^{(R_0-R)\alpha c_0/\beta} |s|\frac{1}{\sigma}\phi(\frac{s}{\sigma})ds. \tag{B-14}$$

Using these expressions we can calculate the derivative dV/dR:

$$\begin{aligned}\frac{dV}{dR} &= -\alpha c_0\Phi(-\frac{1}{\sigma}\frac{\alpha c_0}{\beta}(R_0 - R)) + \alpha c_0(R_0 - R)\frac{\partial}{\partial R}\Phi(-\frac{1}{\sigma}\frac{\alpha c_0}{\beta}(R_0 - R)) + \\ &\quad \frac{\partial}{\partial R}\beta \int_{-\infty}^{(R_0-R)\alpha c_0/\beta} |s|\frac{1}{\sigma}\phi(\frac{s}{\sigma})ds + 1.\end{aligned} \tag{B-15}$$

The second and third terms cancel out because of the continuity in the value of the cost function in s. Thus

$$\frac{dV}{dR} = -\alpha c_0\Phi(-\frac{1}{\sigma}\frac{\alpha c_0}{\beta}(R_0 - R)) + 1. \tag{B-16}$$

This is decreasing in R, which proves the inequality $V'' < 0$ for $R < R_0$.

We now prove the second inequality, $V'' > 0$ for $R > R_0$. Intuitively, increasing R reduces the expected size of operations for $R > R_0$. For a greater R, the marginal reduction in the expected size of operations is smaller because operations are needed less frequently.

Characterizing x is straightforward, given assumption A2, $\beta < \alpha c_0 < 2\beta$. If $R + s \geq R_0$, or equivalently, $s \geq R_0 - R$, then $r(R+s) = r(R) = r_{IOR}$, so the central bank optimally chooses $x = 0$. Otherwise, $\beta < \alpha c_0$ implies that the central bank will set $R + s + x = R_0$, or equivalently, $x = R_0 - R - s$.

Recall that $V(R) = E[\alpha|r(R+s-d+x(R,s)) - r(R)| + \beta|x(R,s)|] + R$. Given the optimal choice of x,

$$E\alpha|r(R+s-d+x(R,s)) - r(R)| = 0. \tag{B-17}$$

$$E\beta|x(R,s)| = \beta \int_{-\infty}^{R_0-R} (R_0 - R - s)\frac{1}{\sigma}\phi(\frac{s}{\sigma})ds. \tag{B-18}$$

Note that $(R_0 - R - s) = |R_0 - R - s|$ given the upper limit of the integral. Given these expressions, we can calculate

$$\frac{dV}{dR} = \beta \int_{-\infty}^{R_0-R} -1\frac{1}{\sigma}\phi(\frac{s}{\sigma})ds + 1 = -\beta\Phi(\frac{1}{\sigma}(R_0 - R)) + 1. \tag{B-19}$$

This expression is increasing in R, which proves the inequality $V'' > 0$ for $R > R_0$.

To prove the third inequality, $V'(R_{LC}) > 0$, we use the following expression derived in the course of proving the first inequality:

$$\frac{dV}{dR} = -\alpha c_0 \Phi(-\frac{1}{\sigma}\frac{\alpha c_0}{\beta}(R_0 - R)) + 1. \tag{B-20}$$

Evaluating this expression for $R = R_{LC}$, we have

$$\frac{dV}{dR}(R_{LC}) = -\alpha c_0 \Phi(-\frac{1}{\sigma}\frac{\alpha c_0}{\beta}U) + 1. \tag{B-21}$$

Given assumption A3, $U/\bar{\sigma} > \Phi^{-1}(1 - 2/(3\alpha c_0))$, along with $\alpha c_0/\beta > 1$ implied by assumption A2, we have

$$\begin{aligned}\frac{dV}{dR}(R_{LC}) &> -\alpha c_0 \Phi(-\Phi^{-1}(1 - \frac{2}{3\alpha c_0})) + 1 = -\alpha c_0[1 - \Phi(\Phi^{-1}(1 - \frac{2}{3\alpha c_0}))] + 1 \\ &= -\alpha c_0[1 - (1 - \frac{2}{3\alpha c_0})] + 1 = \frac{1}{3}.\end{aligned} \tag{B-22}$$

This proves the third inequality.

Similarly, to prove the fourth inequality, $V'(R_0+) < 0$, we use the expression for dV/dR derived for $R > R_0$. Recall that

$$\frac{dV}{dR} = -\beta\Phi(\frac{1}{\sigma}(R_0 - R)) + 1. \tag{B-23}$$

Evaluating this expression at $R = R_0$ gives $dV/dR = -(1/2)\beta + 1$. Assumption A1, $\beta > 2$, implies $dV/dR < 0$.

Using the expression for dV/dR, we see that $\lim_{R\to+\infty}(dV/dR) = 1$, which proves the last inequality, $\lim_{R\to\infty} V'(R) > 0$.

This completes the proof of Proposition 1.

Proof of Proposition 2: The proposition is as follows: Assume that $\rho = 0$ and all model parameters are fixed except for σ. There exists a constant F in $(0, \bar{\sigma}]$ such that if $\sigma < F$, then the scarce-reserve regime is optimal, and if $\sigma > F$, then the ample-reserve regime is optimal.

This proposition is a special case of Proposition 5, which we prove later. Nonetheless working through this proof is helpful in better understanding the result. As explained in Section 3.6, the intuition behind this result is clear—the cost to the central bank loads more heavily on σ in the scarce-reserve regime, thus a larger σ makes the ample-reserve regime more favorable relative to the scarce-reserve regime.

In the course of proving Proposition 1, we derived an explicit formula for $V(\bar{R})$, the cost to the central bank of the scarce-reserve regime:

$$\begin{aligned} V(\bar{R}) &= \alpha c_0 U \Phi(-\frac{1}{\sigma}\frac{\alpha c_0}{\beta}U) + \beta \int_{-\infty}^{U\alpha c_0/\beta} |s|\frac{1}{\sigma}\phi(\frac{s}{\sigma})ds + \bar{R} \\ &= \alpha c_0 U \Phi(-\frac{1}{\sigma}\frac{\alpha c_0}{\beta}U) + \beta\sigma(\sqrt{\frac{2}{\pi}} - \phi(\frac{1}{\sigma}\frac{U\alpha c_0}{\beta})) + \bar{R}. \end{aligned} \tag{B-24}$$

Taking its partial derivative with respect to σ, we have

$$\frac{\partial V(\bar{R})}{\partial \sigma} = \beta(\sqrt{\frac{2}{\pi}} - \phi(\frac{1}{\sigma}\frac{U\alpha c_0}{\beta})). \tag{B-25}$$

Some terms cancel out because of the optimization implicit in the expression for $V(\bar{R})$ in choosing between incurring the cost of interest rate volatility $\alpha c_0 U$ and the cost of operations $\beta|s|$.

We also derived an explicit formula for $V(R_A)$ and $dV/dR(R_A)$:

$$V(R_A) = \beta \int_{-\infty}^{R_0 - R} (R_0 - R - s)\frac{1}{\sigma}\phi(\frac{s}{\sigma})ds + R_A. \tag{B-26}$$

$$\frac{dV}{dR}(R_A) = -\beta\Phi(\frac{1}{\sigma}(R_0 - R)) + 1. \tag{B-27}$$

By solving $dV/dR(R_A) = 0$, we have $R = R_0 + \sigma\Phi^{-1}(1 - (1/\beta))$, an expression we verbally derived in Section 3.6. Using this expression, we can calculate:

$$\begin{aligned}
\frac{\partial V(R_A)}{\partial\sigma} &= \frac{\partial}{\sigma}[\beta\int_{-\infty}^{R_0-R_A}(R_0-R_A-s)\frac{1}{\sigma}\phi(\frac{s}{\sigma})ds+R_A] \\
&= \frac{\partial}{\sigma}[\beta\int_{-\infty}^{(R_0-R_A)/\sigma}(R_0-R_A-t\sigma)\phi(t)dt+R_A] \\
&= \beta\int_{-\infty}^{(R_0-R_A)/\sigma}[-\frac{\partial R_A}{\partial\sigma}-t]\phi(t)dt+\frac{\partial R_A}{\partial\sigma} \\
&= \beta\int_{-\infty}^{(R_0-R_A)/\sigma}-t\phi(t)dt=\beta\phi(\frac{R_0-R_A}{\sigma})=\beta\phi(\Phi^{-1}(\frac{1}{\beta})).
\end{aligned} \tag{B-28}$$

Note that the second line is by a variable replacement, $t = s/\sigma$; the third line is by ignoring the derivative of the upper limit of the integral given that the integrand is zero at the upper limit; and the fourth line is by using the envelope property given that R_A is an optimizer.

Note that $(\partial V/\partial\sigma)(\bar{R}) = \beta(\sqrt{2/\pi} - \phi(U\alpha c_0/(\sigma\beta))) > \beta\sqrt{2/\pi}$ and $(\partial V/\partial\sigma)(R_A) = \beta\phi(\Phi^{-1}(\frac{1}{\beta})) < \beta\sqrt{2/\pi}$. Thus $(\partial V/\partial\sigma)(\bar{R}) > (\partial V/\partial\sigma)(R_A)$; in other words $V(\bar{R}) - V(R_A)$ is strictly increasing in σ.

To prove Proposition 2, we need only to prove $\lim_{\sigma\to 0} V(\bar{R}) - V(R_A) > 0$. This ensures that for small enough σ, a scarce level of supply will always be chosen. This is straightforward, because using formulas for $V(\bar{R})$ and $V(R_A)$ derived previously, we can show that $\lim_{\sigma\to 0} V(\bar{R}) = R_{LC}$ and $\lim_{\sigma\to 0} V(R_A) = R_0$, which is intuitive as the costs due to interest rate volatility and open market operations converge to zero. This completes the proof of Proposition 2.

Note that Proposition 2 allows the possibility that for a given set of parameters, the scarce-reserve regime is always optimal, in which case $F = \bar{\sigma}$. We can show that there exist a 'plenty' of cases where for some σ, the ample-reserve regime is optimal and $F < \bar{\sigma}$. Given that $V(\bar{R}) - V(R_A) = U$ in the limit $\sigma \to 0$, a sufficient condition for the ample-reserve regime to be optimal for some $\sigma \in (0, \bar{\sigma}]$ is that the minimum of $(\partial V/\partial\sigma)(\bar{R}) - (\partial V/\partial\sigma)(R_A)$ is greater than $U/\bar{\sigma}$.

For convenience, suppose that $\bar{\sigma} = U/\Phi^{-1}(1 - 2/(3\alpha c_0))$, the upper limit of $\bar{\sigma}$ consistent with A3. Note that

$$
\begin{aligned}
(\partial V/\partial\sigma)(\bar{R}) - (\partial V/\partial\sigma)(R_A) &= \beta(\sqrt{\frac{2}{\pi}} - \phi(\frac{U\alpha c_0}{\sigma\beta}) - \phi(\Phi^{-1}(\frac{1}{\beta}))) \\
&> \beta(\frac{1}{\sqrt{2\pi}} - \phi(\frac{U\alpha c_0}{\sigma\beta})) \\
&> \beta(\frac{1}{\sqrt{2\pi}} - \phi(\frac{U\alpha c_0}{\bar{\sigma}\beta})) \\
&> \beta(\frac{1}{\sqrt{2\pi}} - \phi(\Phi^{-1}(1 - \frac{2}{3\alpha c_0})) \\
&> \frac{1}{2}\alpha c_0(\frac{1}{\sqrt{2\pi}} - \phi(\Phi^{-1}(1 - \frac{2}{3\alpha c_0}))).
\end{aligned} \tag{B-29}
$$

The last line is implied by assumption A2. We need the last line to be greater than $U/\bar{\sigma}$, which is $\Phi^{-1}(1 - 2/(3\alpha c_0))$. In other words, a sufficient condition for the ample-reserve regime to be optimal for some σ is

$$\frac{1}{2}\alpha c_0(\frac{1}{\sqrt{2\pi}} - \phi(\Phi^{-1}(1 - \frac{2}{3\alpha c_0}))) > \Phi^{-1}(1 - \frac{2}{3\alpha c_0}). \tag{B-30}$$

This condition is easily satisfied because while both sides increase in αc_0, the right-hand side increases extremely slowly in αc_0 for about $\alpha c_0 > 10$, as Φ approaches 1 very fast. For example, imagine evaluating the expression for $\alpha c_0 = 100$; $\Phi^{-1}(1 - \frac{2}{3\alpha c_0})$ is less than 3; even $\Phi^{-1}(1 - 10^{-10})$ is less than 7.

Proof of Proposition 3: The proposition is as follows: Assume $\rho = 0$. $\partial F/\partial\beta \leq 0$. The inequality is strict if F is not at the upper bound, $F < \bar{\sigma}$.

This has almost been proved in the course of proving Proposition 2. Using the expression for $V(\bar{R})$, we calculate

$$\frac{\partial V(\bar{R})}{\partial\beta} = \sigma(\sqrt{\frac{2}{\pi}} - \phi(\frac{1}{\sigma}\frac{U\alpha c_0}{\beta})). \tag{B-31}$$

This is almost identical to the expression for $(\partial V/\partial\sigma)(\bar{R})$ because β and σ both appear only in the form of $\beta\sigma$ in the expression for $V(\bar{R})$. Similarly, using the expression for $V(R_A)$, we calculate

$$\frac{\partial V(R_A)}{\partial\beta} = \sigma\phi(\Phi^{-1}(\frac{1}{\beta})). \tag{B-32}$$

This again is very similar to the expression for $(\partial V/\partial\sigma)(R_A)$. Thus

$$\frac{\partial V(\bar{R})}{\partial\beta} - \frac{\partial V(R_A)}{\partial\beta} = \sigma[\sqrt{\frac{2}{\pi}} - \phi(\frac{1}{\sigma}\frac{U\alpha c_0}{\beta}) - \phi(\Phi^{-1}(\frac{1}{\beta}))] > 0. \tag{B-33}$$

Suppose that $F < \bar{\sigma}$. Then this inequality implies that if β increases by a small amount, the central bank strictly prefers the ample reserves at $\sigma = F$, while it was indifferent prior to the increase in β. Thus $\partial F / \partial \beta < 0$. However, if $F = \bar{\sigma}$, then the inequality might hold not strictly because the central bank might have strictly preferred the scarce-reserve regime prior to the increase in β. This completes the proof of Proposition 3.

Proof of Proposition 4: The proposition is as follows: The cost function $V(R)$ has two local minima: $R = \bar{R}$ (scarce reserves) and another in $R > R_0$ (ample reserves).

This is a generalized version of Proposition 1. The basic ideas underlying the proof are the same, but we rely less on deriving closed-form expressions because doing so is more difficult in this case. As in the proof of Proposition 1, it is sufficient to prove the following five inequalities:

- $V'' < 0$ for $R < R_0$.
- $V'' > 0$ for $R > R_0$.
- $V'(\bar{R}) > 0$.
- $V'(R_0+) < 0$, where $V'(R_0+)$ denotes the right limit of V' at R_0.
- $\lim_{R \to \infty} V'(R) > 0$.

The proof of the first inequality, $V'' < 0$ for $R < R_0$, is based on the idea that increasing R is less costly for R closer to R_0 (while still $R < R_0$) because the flat portion of the demand curve helps lower the cost due to interest rate volatility and central bank operations. We write the central bank's cost function as follows:

$$V(R) = \int_{-\infty}^{+\infty} W(R, s) \mu_s(s) ds + R. \tag{B-34}$$

$W(R, s)$ is the expected cost due to interest rate volatility and central bank operations given (R, s) and μ_s is the probability density function of s; $\mu_s(s) = \phi(s/(\sqrt{1-\rho^2}\sigma))/(\sqrt{1-\rho^2}\sigma$. We can write

$$W(R, s) = \min_x [\beta |x| + \int_{-\infty}^{+\infty} \alpha |r(R + s + x + e) - r(R)| \mu_e(e) de]. \tag{B-35}$$

We denote the demand shock as e instead of d to avoid notational confusion involving integrals. $\mu_e(e)$ is the probability density function of e; $\mu_e(e) = \phi(e/(\sigma\rho))/(\sigma\rho)$. Also, we write $R+s+x+e$ instead of $R+s+x-e$, which we can do because e's distribution is symmetric around zero.

Taking the derivative of $W(R,s)$ with respect to R, we have

$$\begin{aligned}\frac{\partial W}{\partial R} &= \frac{\partial}{\partial R}[\beta|x| + \int_{-\infty}^{+\infty} \alpha|r(R+s+x+e) - r(R)|\mu_e(e)de] \\ &= \int_{-\infty}^{+\infty} \alpha\frac{\partial}{\partial R}|r(R+s+x+e) - r(R)|\mu_e(e)de \\ &= -\alpha c_0 Prob(R+s+x+e \geq R_0) = -\alpha c_0[1 - \Phi(\frac{R_0 - R - s - x}{\sigma\rho})]. \end{aligned} \tag{B-36}$$

In the second line, note that x is not treated as a function of R in calculating the partial derivative. Otherwise, x is implicitly a function of R and s but terms including $\partial x/\partial R$ cancel out because x is either an optimal choice satisfying the first-order condition—envelope property—or $x = 0$ is optimal because moving x to either direction is suboptimal, in which case $\partial x/\partial R = 0$.

Using this expression we can calculate dV/dR and d^2V/dR^2:

$$\frac{dV}{dR} = \int_{-\infty}^{+\infty} -\alpha c_0[1 - \Phi(\frac{R_0 - R - s - x}{\sigma\rho})]\mu_s(s)ds + 1. \tag{B-37}$$

$$\frac{d^2V}{dR^2} = \int_{-\infty}^{+\infty} \alpha c_0 \phi(\frac{R_0 - R - s - x}{\sigma\rho})\frac{1}{\sigma\rho}(-1 - \frac{\partial x}{\partial R})\mu_s(s)ds. \tag{B-38}$$

To show $V'' < 0$, it is sufficient to show that $1 + \partial x/\partial R \geq 0$ for any (R,s). Recall that x is an optimizer of the following problem:

$$\min_x[\beta|x| + \int_{-\infty}^{+\infty} \alpha|r(R+s+x+e) - r(R)|\mu_e(e)de]. \tag{B-39}$$

If $x = 0$, then $\partial x/\partial R = 0$ because moving x in either direction is suboptimal. Therefore,

$1 + \partial x/\partial R = 1 \geq 0$. Suppose that $x \neq 0$. Then, the FOC for x implies

$$0 = \beta sign(x) + \int_{-\infty}^{+\infty} \alpha \frac{\partial}{\partial x} |r(R+s+x+e) - r(R)| \mu_e(e) de$$
$$= \beta sign(x) + \alpha c_0 [-Prob(e < -s - x) + Prob(-s - x < e < -s - x + R_0 - R)]. \quad \text{(B-40)}$$

$sign(x)$ is 1 if $x > 0$ and -1 if $x < 0$. The second line can be understood as follows: if $e < -s - x$, then increasing x moves $r(R+s+x+e)$ toward $r(R)$, reducing the cost. However, if $e > -s - x$, then increasing x moves $r(R+s+x+e)$ away from $r(R)$, except if $e > -s - x + R_0 - R$, in which case there is no change because $r(R+s+x+e)$ is at the flat portion of the demand curve, $r(R+s+x+e) = r_{IOR}$.

Furthermore, since x minimizes the objective function, the partial derivative of the right-hand size with respect to x must be positive:

$$\alpha c_0 [2\mu_e(-s-x) - \mu_e(-s-x+R_0-R)] > 0. \quad \text{(B-41)}$$

Taking the partial derivative of the FOC with respect to R, we have

$$0 = \alpha c_0 [2\mu_e(-s-x) \frac{\partial x}{\partial R} - \mu_e(-s-x+R_0-R)(\frac{\partial x}{\partial R} + 1)]$$
$$= \alpha c_0 [(2\mu_e(-s-x) - \mu_e(-s-x+R_0-R)) \frac{\partial x}{\partial R} - \mu_e(-s-x+R_0-R)]. \quad \text{(B-42)}$$

If $\partial x/\partial R \leq 0$, then the right-hand side of this equation is negative, which is a contradiction. Therefore, $\partial x/\partial R > 0$. This completes the proof of the first inequality, $V'' < 0$ for $R < R_0$.

We now prove the second inequality, $V'' > 0$ for $R > R_0$. This is simpler because we can explicitly characterize x. Since $R > R_0$, the central bank chooses only $x \geq 0$; $x < 0$ is not optimal because it increases the interest rate cost term $|r(R+s+x+e) - r(R)|$. Thus the optimal x satisfies the following FOC, derived earlier:

$$0 = \beta + \int_{-\infty}^{+\infty} \alpha \frac{\partial}{\partial x} |r(R+s+x+e) - r(R)| \mu_e(e) de$$
$$= \beta - \alpha c_0 Prob(e < -s - x + R_0 - R) = \beta - \alpha c_0 \Phi(\frac{-s - x + R_0 - R}{\rho\sigma}). \quad \text{(B-43)}$$

Therefore,

$$x(R, s) = max(y_0 - (R+s), 0); \quad \text{(B-44)}$$
$$y_0 \equiv R_0 - \rho\sigma\Phi^{-1}(\frac{\beta}{\alpha c_0}). \quad \text{(B-45)}$$

Note that by assumption A2, $\beta/(\alpha c_0) > (1/2)$, which implies $y_0 < R_0$.

Recall that $W(R, s)$ is the cost associated with the state s. Using an expression for $\partial W/\partial R$ derived earlier, we have

$$\begin{aligned}\frac{\partial W}{\partial R} &= \int_{-\infty}^{+\infty} \alpha \frac{\partial}{\partial R} |r(R+s+x+e) - r(R)| \mu_e(e) de \\ &= -\alpha c_0 Prob(R+s+x+e \leq R_0) = -\alpha c_0 \Phi(\frac{R_0 - R - s - x}{\rho\sigma}). \end{aligned} \tag{B-46}$$

Note that in the first line, x is not treated as a function of R in calculating the partial derivative.

Therefore,

$$\frac{dV}{dR} = \int_{-\infty}^{+\infty} -\alpha c_0 \Phi(\frac{R_0 - R - s - x}{\rho\sigma}) \mu_s(s) ds + 1. \tag{B-47}$$

$$\frac{d^2 V}{dR^2} = \int_{-\infty}^{+\infty} -\alpha \phi(\frac{R_0 - R - s - x}{\rho\sigma}) \frac{1}{\sigma\rho} (-1 - \frac{\partial x}{\partial R}) \mu_s(s) ds. \tag{B-48}$$

The expression for $x(R, s)$ implies that $\partial x/\partial R$ is either 0 or -1. Therefore, $-1 - (\partial x/\partial R) \leq 0$ and thus $V'' > 0$; the inequality is strict because $\partial x/\partial R = 0$ over the interval $s \in (y_0 - R, +\infty)$ for every (R, s). This proves the second inequality.

We now prove the third inequality, $V'(\bar{R}) > 0$. First, we prove that at $R = \bar{R}$, if $s \leq 0$, then $x + s \leq 0$; x at most offsets s. Suppose that $s \leq 0$ and $x + s > 0$. Then $x > 0$. The FOC for x is

$$0 = \beta - \alpha c_0 Prob(e < -s - x) + \alpha c_0 Prob(-s - x < e < -s - x + R_0 - R). \tag{B-49}$$

$Prob(e < -s - x) < (1/2)$ because $x + s > 0$. Thus the sum of the first two terms alone is greater than $\beta - (1/2)\alpha c_0$, which is positive due to assumption A2, $\beta < \alpha c_0 < 2\beta$. Thus the right-hand side of FOC is positive, which is a contradiction. Therefore, if $x < 0$, then $x + s \leq 0$.

Next, we prove that if $s > 0$, then $x \leq 0$. If $s > 0$ and $x > 0$, then $s + x > 0$, which leads to a contradiction, as we just saw.

Define $\tilde{x}(s) = -s$ if $s \le 0$ and $\tilde{x}(s) = 0$ if $s > 0$. We just showed that for every s, $\tilde{x} \ge x$; we ignore the dependence of x on R because we are looking at $R = \bar{R}$. Then,

$$
\begin{aligned}
\frac{dV}{dR}(\bar{R}) &= \int_{-\infty}^{+\infty} -\alpha c_0[1 - \Phi(\frac{R_0 - \bar{R} - s - x}{\rho\sigma})]\mu_s(s)ds + 1 \\
&\ge \int_{-\infty}^{+\infty} -\alpha c_0[1 - \Phi(\frac{R_0 - \bar{R} - s - \tilde{x}}{\rho\sigma})]\mu_s(s)ds + 1 \\
&= \int_{-\infty}^{0} -\alpha c_0[1 - \Phi(\frac{R_0 - \bar{R}}{\rho\sigma})]\mu_s(s)ds + \int_{0}^{\infty} -\alpha c_0[1 - \Phi(\frac{R_0 - \bar{R} - s}{\rho\sigma})]\mu_s(s)ds + 1 \\
&> -\frac{\alpha c_0}{2}[1 - \Phi(\frac{R_0 - \bar{R}}{\sigma})] + \int_{-\infty}^{\infty} -\alpha c_0[1 - \Phi(\frac{R_0 - \bar{R} - s}{\rho\sigma})]\mu_s(s)ds + 1 \\
&= -\frac{\alpha c_0}{2}[1 - \Phi(\frac{R_0 - \bar{R}}{\sigma})] + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\alpha c_0 I(\bar{R} + s + e > R_0)\mu_e(e)\mu_s(s)deds + 1 \\
&= -\frac{\alpha c_0}{2}[1 - \Phi(\frac{R_0 - \bar{R}}{\sigma})] - \alpha c_0[1 - \Phi(\frac{R_0 - \bar{R}}{\sigma})] + 1 \\
&= -\frac{3\alpha c_0}{2}[1 - \Phi(\frac{R_0 - \bar{R}}{\sigma})] + 1 > 0. \qquad \text{(B-50)}
\end{aligned}
$$

The second line is a direct result of $\tilde{x} \ge x$. In the third line, we are putting in actual values of $\tilde{x}$ into the equation. In the fourth line, the first term follows from the fact that $0 < \rho < 1$, thus removing ρ decreases the argument of the function Φ. The integral in the second term has its range of integration expanded. In the fifth line, the integrand is written as an integral over the indicator function I, which takes the value of 1 if the inequality inside its argument is satisfied and 0 otherwise. In the sixth line, we use the fact that $s + e$ follows a normal distribution with mean 0 and standard deviation σ. In the last line, we use assumption A3, $U/\sigma = (R_0 - \bar{R})/\sigma > \Phi^{-1}(1 - 2/(3\alpha c_0))$. These lines of equations and inequalities prove the third inequality, $dV/dR(\bar{R}) > 0$.

Next, we prove the fourth inequality, $V'(R_0+) < 0$. We showed earlier that at $R = R_0$, $x = max(y_0 - R_0 - s, 0)$, where $y_0 = R_0 - \rho\sigma\Phi^{-1}(\beta/(\alpha c_0))$. Using the expression for (dV/dR)

derived earlier for $R \geq R_0$, we write

$$
\begin{aligned}
\frac{dV}{dR}(R_0) &= \int_{-\infty}^{+\infty} -\alpha c_0 \Phi(\frac{-s-x}{\rho\sigma})\mu_s(s)ds + 1 \\
&= \int_{-\infty}^{y_0-R_0} -\alpha c_0 \Phi(\frac{R_0-y_0}{\rho\sigma})\mu_s(s)ds + \int_{y_0-R_0}^{+\infty} -\alpha c_0 \Phi(\frac{-s}{\rho\sigma})\mu_s(s)ds + 1 \\
&= \int_{-\infty}^{y_0-R_0} -\beta\mu_s(s)ds + \int_{y_0-R_0}^{+\infty} -\alpha c_0 \Phi(\frac{-s}{\rho\sigma})\mu_s(s)ds + 1 \\
&< \int_{-\infty}^{y_0-R_0} -\beta\Phi(\frac{-s}{\rho\sigma})\mu_s(s)ds + \int_{y_0-R_0}^{+\infty} -\beta\Phi(\frac{-s}{\rho\sigma})\mu_s(s)ds + 1 \\
&= \int_{-\infty}^{+\infty} -\beta\Phi(\frac{-s}{\rho\sigma})\mu_s(s)ds + 1 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} -\beta I(s+e<0)\mu_e(e)\mu_s(s)deds + 1 \\
&= -\frac{1}{2}\beta + 1 < 0.
\end{aligned} \tag{B-51}
$$

The second line directly follows from the explicit form of x as a function of s. The third line substitutes y_0 in the integrand by $R_0 - \rho\sigma\Phi^{-1}(\beta/(\alpha c_0))$. In the fourth line, inserting Φ makes the first term larger and replacing αc_0 by β makes the second term larger, given assumption A2, $\beta < \alpha c_0 < 2\beta$. In the fifth line, the two integrals are combinded into one. In the sixth line, Φ is replaced by integration over e. The last line follows from the fact that $s + e$ is normally distributed with mean zero, and from assumption A1, $\beta > 2$. This proves $V'(R_0+) < 0$.

Lastly, we prove $\lim_{R\to\infty} V'(R) > 0$.

$$
\begin{aligned}
\frac{dV}{dR} &= \int_{-\infty}^{+\infty} -\alpha c_0 \Phi(\frac{R_0-R-s-x}{\rho\sigma})\mu_s(s)ds + 1 \\
&\geq \int_{-\infty}^{+\infty} -\alpha c_0 \Phi(\frac{R_0-R-s}{\rho\sigma})\mu_s(s)ds + 1 \\
&= -\alpha c_0 Prob(s+e<R_0-R) + 1.
\end{aligned} \tag{B-52}
$$

The second line follows from $x \geq 0$, as derived earlier. The third line is derived by writing down Φ as the probability involving e and integrating over s; equivalently, we can derive it by writing Φ as the integration of an indicator function over e, as we did multiple times earlier. The last line converges to 1 as $R \to \infty$ because $s + e$ follows a normal distribution.

This completes the proof of Proposition 4.

Proof of Proposition 5: The proposition is as follows: Assume that all model parameters are fixed except for σ and ρ. There exists a function $F(\rho) \in (0, \bar{\sigma}]$ such that if $\sigma < F(\rho)$, then the scarce-reserve regime is optimal, and if $\sigma > F(\rho)$, then the ample-reserve regime is optimal. F is monotonically decreasing in ρ.

This is a generalization of Proposition 2.

First, we prove the first part of the proposition that a larger σ makes ample reserve preferable holding everything else constant, including ρ: There exists a function $F(\rho) \in (0, \bar{\sigma}]$ such that if $\sigma < F(\rho)$, then the scarce-reserve regime is optimal, and if $\sigma > F(\rho)$, then the ample-reserve regime is optimal.

It is convenient to introduce an additional parameter γ in the central bank's cost function in the form of γR:

$$V(R) = \min_{x(R,s)} E[\alpha|r(R+s+x-e) - r(R)| + \beta|x|] + \gamma R. \tag{B-53}$$

In addition, for convenience, we assume $R_{LC} = 0$ without loss of generality. Then, $R_0 = U$. Therefore, the cost function can be written as

$$\begin{aligned} V(R) &= \min_x E[\alpha c_0 |min(s+x-e, U)| + \beta|x|] + \gamma R \text{ if } R \leq U; \\ &= \min_x E[\alpha c_0 |min(s+x-e+R-U, 0)| + \beta|x|] + \gamma R \text{ if } R > U. \end{aligned} \tag{B-54}$$

V is a function of R as well as model parameters, so we write $V(R, \alpha, \beta, \gamma, \sigma, U)$ whenever it is helpful to explicitly state the dependence on parameters. We omit the other parameters, c_0 and ρ, because they are always fixed while we prove the first part of the proposition.

Consider a baseline vector of parameters satisfying the assumptions A1, A2 and A3: $P_{BS} \equiv (\alpha, \beta, \gamma = 1, \sigma, U)$. Consider an alternative vector of identical parameters except for σ: $P_{ALT} = (\alpha, \beta, 1, \sigma', U)$, where $\sigma' > \sigma$. Define $\eta \equiv \sigma'/\sigma$ and $P_{ALT} = (\alpha, \beta, 1, \eta\sigma, U)$.

Similarly, the level of ample reserves can be written as a function of parameters, $R_A(\alpha, \beta, \gamma, \sigma, U)$. Formally, $R_A = argmin_{R \geq R_0} V(R)$, whose existence is guaranteed by Proposition 4, as long as the assumptions A1 through A3 are satisfied with α and β replaced by α/γ and β/γ. This is because scaling the cost function of the central bank by γ results in identical optimizing choices.

To prove the first proposition, it is sufficient to show the following:

$$\frac{d}{d\sigma}[V(R_A(\alpha, \beta, 1, \sigma, U), \alpha, \beta, 1, \sigma, U) - V(0, \alpha, \beta, 1, \sigma, U)] < 0 \tag{B-55}$$

if $V(R_A(P_{BS}, P_{BS}) - V(0, P_{BS}) = 0$. This is intuitive because the inequality implies that increasing σ by a small amount makes ample reserves more favorable for parameter values at the border of the region over which ample reserves are more favorable.

Formally, suppose that for given parameter values (other than σ), ample reserves are more favorable for some σ, $V(R_A) \leq V(0)$; let S denote the set of such σ. To prove the first part of the proposition, it is enough to show that if ample reserves are preferred, $V(R_A) - V(0) \leq 0$, for some $\sigma = a$, then ample reserves are preferred for every $\sigma > a$. Suppose that this is not the case: $V(R_A) - V(0) \leq 0$ for $\sigma = a$ but $V(R_A) - V(0) > 0$ for $\sigma = b$, where $b > a$. Let B be the set of σ such that $V(R_A) - V(0) > 0$ and $\sigma > a$, which is not empty by assumption. Suppose that $\inf B > a$. Since V and R_A are continuous in σ, $V(R_A) - V(0) = 0$ at $\sigma = \inf B$. Then, if the stated inequality, $(d/d\sigma)(V(R_A) - V(0)) < 0$, holds at $\sigma = \inf B$, there exists a' such that $a < a' < \inf B$ and $V(R_A) - V(0) > 0$ at $\sigma = a'$, which implies $a' \geq \inf B$, a contradiction. Now suppose that $\inf B = a$. Since $V(R_A) - V(0) > 0$ for every $\sigma \in B$ and $V(R_A) - V(0) \leq 0$ at $\sigma = a$, the continuity of $V(R_A) - V(0)$ implies $V(R_A) - V(0)$ at $\sigma = a$. If the inequality $(d/d\sigma)(V(R_A) - V(0)) < 0$ holds at $\sigma = a$, then there exists $\delta > 0$ such that $V(R_A) - V(0) < 0$ for every $\sigma \in (a, a + \delta)$, which implies $\inf B \geq a + \delta$, a contradiction.

Note that $V(0, \alpha, \beta, 1, \sigma', U) = V(0, \alpha, \beta, 1, \eta\sigma, U) = V(0, \eta\alpha, \eta\beta, \eta, \sigma, U/\eta)$. To see this,

$$\begin{aligned}
V(0, \alpha, \beta, 1, \eta\sigma, U) &= \min_x E[\alpha c_0 |min(s + x - e, U)| + \beta|x|] \\
&= \min_x E[\eta\alpha c_0 |min(\frac{s}{\eta} + \frac{x}{\eta} - \frac{e}{\eta}, \frac{U}{\eta})| + \eta\beta|\frac{x}{\eta}|] \\
&= \min_x E[\eta\alpha c_0 |min(\frac{s}{\eta} + x - \frac{e}{\eta}, \frac{U}{\eta})| + \eta\beta|x|] = V(0, \eta\alpha, \eta\beta, \eta, \sigma, \frac{U}{\eta}).
\end{aligned} \tag{B-56}$$

From the second to the third line, x/η is simply replaced by x because we can treat the

x/η as the optimizing choice. The last equality holds because $s/\eta \sim \mathcal{N}(0,(1-\rho^2)\sigma^2)$ and $e/\eta \sim \mathcal{N}(0,\rho^2\sigma^2)$.

Similarly, $V(R_A(\alpha,\beta,1,\eta\sigma,U),\alpha,\beta,1,\eta\sigma,U) = V(R_A(\eta\alpha,\eta\beta,\sigma,\eta,U/\eta),\eta\alpha,\eta\beta,\sigma,\eta,U/\eta)$. To see this (we define V_A as the value of V at the optimally chosen R_A for conciseness),

$$\begin{aligned}
V_A(\alpha,\beta,1,\eta\sigma,U) &= \min_{R\geq U,x} E[\alpha c_0|min(s+x-e+R-U,0)|+\beta|x|]+R \\
&= \min_{R/\eta\geq U/\eta,x} E[\eta\alpha c_0|min(\frac{s}{\eta}+\frac{x}{\eta}-\frac{e}{\eta}+\frac{R}{\eta}-\frac{U}{\eta},0)|+\eta\beta|\frac{x}{\eta}|]+\eta\frac{R}{\eta} \\
&= \min_{R\geq U/\eta,x} E[\eta\alpha c_0|min(\frac{s}{\eta}+x-\frac{e}{\eta}+R-\frac{U}{\eta},0)|+\eta\beta|x|]+\eta R \\
&= V_A(\eta\alpha,\eta\beta,\eta,\sigma,\frac{U}{\eta}).
\end{aligned} \tag{B-57}$$

In addition, $\eta V(0,\alpha,\beta,1,\sigma,U) = V(0,\eta\alpha,\eta\beta,\eta,\sigma,U)$. This can be simply seen by multiplying the cost function of the central bank by η. Similarly $\eta V_A(\alpha,\beta,1,\sigma,U) = V_A(\eta\alpha,\eta\beta,\eta,\sigma,U)$.

Next, we calculate $V_A(\alpha,\beta,1,\sigma',U) - V_A(\alpha,\beta,1,\sigma,U)$:

$$\begin{aligned}
&V_A(\alpha,\beta,1,\sigma',U) - V_A(\alpha,\beta,1,\sigma,U) = \\
&\quad V_A(\eta\alpha,\eta\beta,\eta,\sigma,\frac{U}{\eta}) - \frac{1}{\eta}V_A(\eta\alpha,\eta\beta,\eta,\sigma,U).
\end{aligned} \tag{B-58}$$

Note that $V_A(\eta\alpha,\eta\beta,\eta,\sigma,U) = V_A(\eta\alpha,\eta\beta,\eta,\sigma,U/\eta) + \eta(U-U/\eta)$. To see this (define $\Delta \equiv U - U/\eta$ for convenience),

$$\begin{aligned}
V_A(\eta\alpha,\eta\beta,\eta,\sigma,U) &= \min_{R\geq U,x} E[\eta\alpha c_0|min(s+x-e+R-U,0)|+\eta\beta|x|]+\eta R \\
&= \min_{R-\Delta/\geq U-\Delta,x} E[\eta\alpha c_0|min(s+x-e+(R-\Delta)-(U-\Delta),0)|+\eta\beta|x|]+\eta(R-\Delta)+\eta\Delta \\
&= V_A(\eta\alpha,\eta\beta,\eta,\sigma,\frac{U}{\eta})+\eta(U-\frac{U}{\eta}).
\end{aligned} \tag{B-59}$$

The last line follows from treating $R-\Delta$ as the optimizer.

Thus, we can further simplify the expression for $V_A(\alpha,\beta,1,\sigma',U) - V_A(\alpha,\beta,1,\sigma,U)$:

$$V_A(\alpha,\beta,1,\sigma',U) - V_A(\alpha,\beta,1,\sigma,U) = (1-\frac{1}{\eta})V_A(\eta\alpha,\eta\beta,\eta,\sigma,U) - \eta(1-\frac{1}{\eta})U. \tag{B-60}$$

We divide this expression by $\sigma'-\sigma$ and take the limit $\sigma'\to\sigma$ to calculate the following derivative:

$$\frac{d}{d\sigma}V_A(\alpha,\beta,1,\sigma,U) = \lim_{\sigma'\to\sigma}\frac{1}{\sigma'-\sigma}[V_A(\alpha,\beta,1,\sigma',U) - V_A(\alpha,\beta,1,\sigma,U)]$$
$$= \frac{1}{\sigma}\lim_{\eta\to 1}\frac{1}{\eta-1}[(1-\frac{1}{\eta})V_A(\eta\alpha,\eta\beta,\eta,\sigma,U) - \eta(1-\frac{1}{\eta})U]$$
$$= \frac{1}{\sigma}[V_A(\alpha,\beta,1,\sigma,U) - U]. \tag{B-61}$$

Similarly, we calculate the following derivative:

$$\frac{d}{d\sigma}V(0,\alpha,\beta,1,\sigma,U) = \frac{1}{\sigma}\lim_{\eta\to 1}\frac{1}{\eta-1}[(1-\frac{1}{\eta})V(0,\eta\alpha,\eta\beta,\eta,\sigma,U)$$
$$+ V(0,\eta\alpha,\eta\beta,\eta,\sigma,\frac{U}{\eta}) - V(0,\eta\alpha,\eta\beta,\eta,\sigma,U)]$$
$$= \frac{1}{\sigma}[V(0,\alpha,\beta,1,\sigma,U) + [\frac{d}{d\eta}\frac{U}{\eta}]_{\eta=1}\frac{\partial}{\partial U}V(0,\alpha,\beta,1,\sigma,U)]$$
$$= \frac{1}{\sigma}[V(0,\alpha,\beta,1,\sigma,U) - U\frac{\partial}{\partial U}V(0,\alpha,\beta,1,\sigma,U)] \tag{B-62}$$

Therefore,

$$\frac{d}{d\sigma}[V_A(\alpha,\beta,1,\sigma,U) - V(0,\alpha,\beta,1,\sigma,U)]$$
$$= \frac{1}{\sigma}[V_A(\alpha,\beta,1,\sigma,U) - V(0,\alpha,\beta,1,\sigma,U)] - \frac{U}{\sigma}[1 - \frac{\partial}{\partial U}V(0,\alpha,\beta,1,\sigma,U)]. \tag{B-63}$$

We need to show that this quantity is negative if $V_A(\alpha,\beta,1,\sigma,U) - V(0,\alpha,\beta,1,\sigma,U)$ is zero. The first term in the equation drops out if $V_A(\alpha,\beta,1,\sigma,U) - V(0,\alpha,\beta,1,\sigma,U) = 0$, so it is sufficient to show that $(\partial/\partial U)V(0,\alpha,\beta,1,\sigma,U) < 1$.

We explicitly write down the form of V:

$$V(0,\alpha,\beta,1,\sigma,U) = \int_{s+x(x,U)-U}^{+\infty}\int_{-\infty}^{+\infty}\alpha c_0|s+x(s,U)-e|\mu_s(s)\mu_e(e)dsde$$
$$+ \int_{-\infty}^{s+x(x,U)-U}\int_{-\infty}^{+\infty}\alpha c_0 U\mu_s(s)\mu_e(e)dsde + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\beta|x(s,U)|\mu_s(s)\mu_e(e)dsde. \tag{B-64}$$

As before, μ_s and μ_e are probability density functions of s and e, respectively, and we make the dependence of x on s and U explicit.

Differentiating with respect to U, we have

$$\frac{\partial}{\partial U}V(0,\alpha,\beta,1,\sigma,U) = \int_{-\infty}^{s+x(x,U)-U}\int_{-\infty}^{+\infty}\alpha c_0\mu_s(s)\mu_e(e)dsde = \alpha c_0 Prob(s+x-e \geq U). \tag{B-65}$$

Integration of terms containing the derivative of x drops out due to the envelope property.

We showed that $\alpha c_0 Prob(s + x - e \geq U) < 1$ in proving Proposition 4; see Equation B-50. Essentially, we showed that

$$\alpha c_0 Prob(s + x - e \geq U) \leq \frac{3\alpha c_0}{2} Prob(s - e \geq U) < 1. \tag{B-66}$$

The right-hand side inequality follows from assumption A3.

This completes the proof of the first part of the proposition.

Next, we prove the second part of the proposition: F is monotonically decreasing in ρ. As before, let $V(R, \sigma_s, \sigma_e)$ denote the central bank's cost. Since we are interested in changing ρ only, we express V as a function of the supply and the demand shocks' standard deviations, σ_s and σ_e, respectively, as well as the baseline reserve supply. By definition,

$$\sigma_S = \sqrt{1 - \rho^2}\sigma; \text{ and} \tag{B-67}$$

$$\sigma_E = \rho\sigma. \tag{B-68}$$

To prove that F is monotonically decreasing in ρ, it is sufficient to prove

$$\frac{d}{d\rho} V(R_A(\sigma_s, \sigma_e), \sigma_s, \sigma_e) - \frac{d}{d\rho} V(0, \sigma_s, \sigma_e) < 0. \tag{B-69}$$

Note that R_A is the optimal level of ample reserves, which minimizes V in $R \geq U$. As before, we normalize $R_{LC} = 0$. To see that proving this inequality is sufficient, note that $F(\rho)$ is implicitly defined as $V(R_A(\sigma_s, \sigma_e), \sigma_s, \sigma_e) - V(0, \sigma_s, \sigma_e) = 0$, with $\sigma = F(\rho)$. Differentiating the equation with respect to ρ, we have

$$\frac{dX}{d\rho} + \frac{dX}{d\sigma}\frac{dF}{d\rho} = 0. \tag{B-70}$$

$$\frac{dF}{d\rho} = -(\frac{dX}{d\sigma})^{-1}\frac{dX}{d\rho}. \tag{B-71}$$

where $X = V(R_A(\sigma_s, \sigma_e), \sigma_s, \sigma_e) - V(0, \sigma_s, \sigma_e)$. Previously we showed that $dX/d\sigma < 0$ if $X = 0$. Therefore, if $dX/d\rho < 0$, then $dF/d\rho < 0$, which proves the second part of the proposition.

We first characterize $V(0,\sigma_s,\sigma_e)$. We can write

$$V(0,\sigma_s,\sigma_e) = \int_{-\infty}^{\infty} Z(s,\sigma_e)\mu_s(s)ds. \tag{B-72}$$

$$Z(s,\sigma_e) = \int_{-\infty}^{\infty} \alpha c_0|min(s+x(s,\sigma_e)-e,U)|\mu_e(e)de + \beta|x(s,\sigma_e)|, \tag{B-73}$$

where $x(s,\sigma_e)$ is the optimal choice of x given s and σ_e.

Next, we calculate $\partial V/\partial\sigma_s$ and $\partial V/\partial\sigma_e$. Using integration by parts, we write

$$\begin{aligned}
\frac{\partial}{\partial\sigma_s}V(0,\sigma_s,\sigma_e) &= \frac{\partial}{\partial\sigma_s}\int_{-\infty}^{\infty} Z(s,\sigma_e)\mu_s(s)ds = \int_{-\infty}^{\infty} Z\frac{\partial\mu_s}{\partial\sigma_s}ds = \int_{-\infty}^{\infty} Z\mu_s(-\frac{1}{\sigma_s}+\frac{s^2}{\sigma_s^3})ds \\
&= \left[-\frac{Z\mu_s s}{\sigma_s}\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\frac{\partial Z}{\partial s}\mu_s\frac{s}{\sigma_s}ds \\
&= \left[-\frac{Z\mu_s s}{\sigma_s}\right]_{-\infty}^{\infty} + \left[-\frac{\partial Z}{\partial s}\mu_s\sigma_s\right]_{-\infty}^{\infty} + \sigma_s\int_{-\infty}^{\infty}\frac{\partial^2 Z}{\partial s^2}\mu_s ds \\
&= \sigma_s\int_{-\infty}^{\infty}\frac{\partial^2 Z}{\partial s^2}\mu_s ds.
\end{aligned} \tag{B-74}$$

Given the form of Z, Z and Z' will grow at most linearly with s, while μ_s is proportional to $exp(-s^2/(2\sigma_s^2))$. Therefore, the expressions inside the two brackets converge to zero for $s\to\infty$ and $s\to-\infty$.

To characterize $\partial^2 Z/\partial s^2$, we characterize $x(s,\sigma_e)$ first. Define

$$W(s,x,\sigma_e) = \int_{-\infty}^{\infty} \alpha c_0|min(s+x-e,U)|\mu_e de. \tag{B-75}$$

Here x is used as a variable while $x(s,\sigma_e)$ in the expression for Z is the optimal choice of x. Then,

$$Z(s,\sigma_e) = W(s,x(s,\sigma_e),\sigma_e) + \beta|x(s,\sigma_e)|. \tag{B-76}$$

Note that

$$\frac{\partial W}{\partial x} = \alpha c_0[-\Phi(-\frac{s+x}{\sigma_e}) + \Phi(\frac{s+x}{\sigma_e}) - \Phi(\frac{s+x}{\sigma_e}-\frac{U}{\sigma_e})]. \tag{B-77}$$

For $s \leq 0$, $\partial W/\partial X$ increases monotonically in x if $s+x \leq 0$, or equivalently $x \leq -s$. To see this, note that

$$\frac{\partial^2 W}{\partial x^2} = \alpha c_0[\mu_e(-s-x) + \mu_e(s+x) - \mu_e(s+x-U)], \tag{B-78}$$

which is positive if $s+x \leq 0$. Also, note that $\partial W/\partial x \to -\alpha c_0 < -\beta$ as $x \to -\infty$ (assumption A2), and for $x = -s$, $\partial W/\partial x > -\beta$. To see this, with $s + x = 0$,

$$\frac{\partial W}{\partial x} = -\alpha c_0 \Phi(-\frac{U}{\sigma_e}) \geq -\alpha c_0 \Phi(-\Phi^{-1}(1 - \frac{2}{3\alpha c_0})) = -\frac{2}{3} > -\beta. \tag{B-79}$$

For convenience, define $s_1 < 0$ such that $\partial W(s_1, 0, \sigma_e)/\partial x = -\beta$, which is unique given the preceding characterization of $\partial W/\partial x$. Then, the optimal choice of x for $s \leq 0$ is $x = 0$ if $s_1 \leq s \leq 0$ and $x = s_1 - s$ if $s \leq s_1$; note that this depends on the behavior of $\partial W/\partial x$ for $s + x > 0$, and we will shortly show that $\partial W/\partial x > -\beta$ for $s + x > 0$, which is sufficient.

Next, we characterize the optimal choice of x for $s \geq 0$. First, note that $\partial W/\partial x$ is concave in x if $0 \leq s + x \leq U$ (equivalently $-s \leq x \leq -s + U$). To see this, recall that $\partial^2 W/\partial x^2 = \alpha c_0[\mu_e(-s-x) + \mu_e(s+x) - \mu_e(s+x-U)]$ is monotonically decreasing in x over $0 \leq s + x \leq U$. Note that we already showed that for $x = -s$, $\partial W/\partial x > -\beta$. For $x \geq -s + U$,

$$\begin{aligned} |\frac{\partial W}{\partial x}| &= \alpha c_0| - \Phi(-\frac{s+x}{\sigma_e}) + \Phi(\frac{s+x}{\sigma_e}) - \Phi(\frac{s+x}{\sigma_e} - \frac{U}{\sigma_e})| \\ &\leq \alpha c_0[|\Phi(-\frac{s+x}{\sigma_e})| + |\Phi(\frac{s+x}{\sigma_e}) - \Phi(\frac{s+x}{\sigma_e} - \frac{U}{\sigma_e})|]. \end{aligned} \tag{B-80}$$

Note that $|\Phi(-(s+x)/\sigma_e)| \leq \Phi(-U/\sigma_e)$ and $|\Phi((s+x)/\sigma_e) - \Phi((s+x-U)/\sigma_e)| \leq \Phi(U/\sigma_e) - (1/2)$, given $s + x \geq U$. Therefore,

$$|\frac{\partial W}{\partial x}| \leq \alpha c_0[\Phi(-\frac{U}{\sigma_e}) + \Phi(\frac{U}{\sigma_e}) - \frac{1}{2}] = \frac{\alpha c_0}{2} < \beta. \tag{B-81}$$

Next, let $s_2 > 0$ be the smaller of the two solutions to $\partial W(s_2, 0, \sigma_e)/\partial x = \beta$. Given the shape of $\partial W/\partial x$, if the maximum of $\partial W(s, 0, \sigma_e)/\partial x$ over $0 \leq s \leq U$ is greater than β, then there will be two solutions to the equation defining s_2. If not, there will be one or zero solution and $x = 0$ will be optimal for all $s \geq 0$. For now, let us assume that there will be two solutions to the equation defining s_2. We will later see that this assumption makes no difference in proving the proposition.

Given this characterization, the optimal choice of x for $s \geq 0$ is $x = 0$ if $s \leq s_2$ or $s \geq s_3$; and $x = s_2 - s$ if $s_2 \leq s \leq s_3$. s_3 is the maximum value of s for which it is beneficial to choose $x = s_2 - s$, defined as the solution to the following equation:

$$W(s_3, 0, \sigma_e) = W(s_3, s_2 - s_3, \sigma_e) + \beta|s_2 - s_3| = W(s_2, 0, \sigma_e) + \beta|s_2 - s_3|. \qquad \text{(B-82)}$$

The solution s_3 is unique given the shape of $\partial W/\partial x$. In particular, it is greater than the larger solution to the equation defining s_2.

Two things are worth noting. First, $\partial W(s_3, 0, \sigma_e)/\partial x < \beta$, which will be used later. Also, note that $\partial W(s, 0, \sigma_e)/\partial x > -\beta$ for all $s \geq 0$ given the characterization of $\partial W/\partial x$, which validates the characterization of optimal x for $s \leq 0$ discussed earlier.

Given the characterization of the optimal choice of x, we can fully characterize $\partial^2 Z/\partial s^2$. Note that since W depends on s and x through $s + x$ only, we can differentiate W with respect to s and x interchangeably. First, note that due to a discrete jump in $\partial Z/\partial s$ at s_3, we need a term including the Dirac delta function, $(\partial W(s_3, 0, \sigma_e)/\partial s - \beta)\delta(s - s_3)$.[29] As is standard, $\delta(s - s_3)$ is zero if $s \neq s_3$ and becomes 1 if integrated over an interval including a neighborhood of s_3. The remaining expression is based on the fact that $s + x(s, \sigma_e)$ is constant over the regions $s \leq s_1$ and $s_2 \leq s < s_3$:

$$\begin{aligned}\frac{\partial^2 Z}{\partial s^2} &= [\frac{\partial W(s_3, 0, \sigma_e)}{\partial s} - \beta]\delta(s - s_3) \text{ if } s < s_1 \text{ or } s_2 < s < s_3;\\ &= \frac{\partial^2 W(s, 0, \sigma_e)}{\partial s^2} + [\frac{\partial W(s_3, 0, \sigma_e)}{\partial s} - \beta]\delta(s - s_3) \text{ if } s_1 < s < s_2 \text{ or } s > s_3. \qquad \text{(B-83)}\end{aligned}$$

Note that $\partial W(s_3, 0, \sigma_e)/\partial s - \beta < 0$, as discussed earlier.

Similarly we characterize $\partial V/\partial \sigma_e$:

$$\begin{aligned}\frac{\partial}{\partial \sigma_e} V(0, \sigma_s, \sigma_e) &= \frac{\partial}{\partial \sigma_e} \int_{-\infty}^{\infty} [W(s, x(s, \sigma_e), \sigma_e) + \beta|x(s, \sigma_e)|]\mu_s(s) ds\\ &= \int_{-\infty}^{\infty} \frac{\partial}{\partial \sigma_e} W(s, x(s, \sigma_e), \sigma_e)\mu_s ds\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha c_0 |min(s + x(s, \sigma_e) - e, U)| \frac{\partial \mu_e}{\partial \sigma_e} de \mu_s ds\\ &= \sigma_e \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \alpha c_0 \frac{\partial^2}{\partial e^2} |min(s + x(s, \sigma_e) - e, U)| \mu_e de \mu_s ds. \qquad \text{(B-84)}\end{aligned}$$

[29] We can avoid using the delta function by breaking down the second integration by parts into two different intervals to have an explicit term taking into account the jump in $\partial Z/\partial S$, in deriving the expression for $\partial V/\partial \sigma_s$. Using the delta function is more convenient.

The second line follows from the envelope property given that $x(s, \sigma_e)$ is an optimal choice, the third line from the definition of W, and the last line from using integration by parts twice as earlier.

Notice that

$$\frac{\partial^2}{\partial e^2}|min(s + x - e, U)| = 2\delta(e - s - x) - \delta(e - s - x + U). \tag{B-85}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \alpha c_0 \frac{\partial^2}{\partial e^2}|min(s + x(s, \sigma_e) - e, U)|\mu_e de &= \alpha c_0[2\mu_e(s + x) - \mu_e(s + x - U)] \\ &= \frac{\partial^2 W(s, x(s, \sigma_e), \sigma_e)}{\partial s^2}.\end{aligned} \tag{B-86}$$

This implies

$$\frac{\partial}{\partial \sigma_e} V(0, \sigma_s, \sigma_e) = \sigma_e \int_{-\infty}^{\infty} \frac{\partial^2 W(s, x(s, \sigma_e), \sigma_e)}{\partial s^2} \mu_s ds. \tag{B-87}$$

Next we characterize $dV(0, \sigma_s, \sigma_e)/d\rho$:

$$\begin{aligned}\frac{d}{d\rho} V(0, \sigma_s, \sigma_e) &= -\frac{\rho\sigma}{\sqrt{1 - \rho^2}} \frac{\partial}{\partial \sigma_s} V(0, \sigma_s, \sigma_e) + \sigma \frac{\partial}{\partial \sigma_e} V(0, \sigma_s, \sigma_e) \\ &= \rho\sigma^2 \int_{-\infty}^{\infty} [-\frac{\partial^2 Z}{\partial s^2} + \frac{\partial^2 W}{\partial s^2}]\mu_s ds \\ &= \rho\sigma^2 \int_{s \leq s_1, s_2 \leq s \leq s_3} \frac{\partial^2 W}{\partial s^2} \mu_s ds + [\beta - \frac{\partial W(s_3, 0, \sigma_e)}{\partial s}]\mu_s(s_3) \\ &= \rho\sigma^2 [\Phi(\frac{s_1}{\sigma_s}) \frac{\partial^2 W(s_1, 0, \sigma_e)}{\partial s^2} + (\Phi(\frac{s_3}{\sigma_s}) - \Phi(\frac{s_2}{\sigma_s})) \frac{\partial^2 W(s_2, 0, \sigma_e)}{\partial s^2}] \\ &+ [\beta - \frac{\partial W(s_3, 0, \sigma_e)}{\partial s}]\mu_s(s_3) \\ &> \rho\sigma^2 [\Phi(\frac{s_1}{\sigma_s}) \frac{\partial^2 W(s_1, 0, \sigma_e)}{\partial s^2} + (\Phi(\frac{s_3}{\sigma_s}) - \Phi(\frac{s_2}{\sigma_s})) \frac{\partial^2 W(s_2, 0, \sigma_e)}{\partial s^2}].\end{aligned} \tag{B-88}$$

Similarly we can characterize $dV(R_A, \sigma_s, \sigma_e)/d\rho$. There are two differences between this case and the scarce reserve case. First, the level of ample reserves, R_A, is a function of σ_s and σ_e but the terms containing differentials of R_A cancel out because R_A itself is an optimizer in the region $R \geq U$. Second, the form of W changes slightly and depends on R_A. Formally,

$$\begin{aligned}
W_A(s, x, \sigma_e, R) &= \int_{-\infty}^{\infty} \alpha c_0 |min(s + x + R - U - e, 0)| \mu_e de; \\
Z_A(s, \sigma_e, R) &= W_A(s, x(s, \sigma_e, R), \sigma_e, R) + \beta |x(s, \sigma_e, R)|; \\
V(R, \sigma_s, \sigma_e) &= \int_{-\infty}^{\infty} Z_A(s, \sigma, R_A(\sigma_s, \sigma_e)) \mu_s ds + R. \qquad \text{(B-89)}
\end{aligned}$$

Following the steps outlined earlier, we can derive the following expression:

$$\frac{d}{d\rho} V(R_A(\sigma_s, \sigma_e), \sigma_s, \sigma_e) = \rho\sigma^2 \Phi(\frac{s_A}{\sigma_s}) \frac{\partial^2 W_A(s_A, 0, \sigma_e, R_A)}{\partial s^2}. \qquad \text{(B-90)}$$

This expression follows from the fact that there exists $s_A < -R_A + U$ such that if $s \leq s_A$, then the optimal $x = s_A - s$, and if $s > s_A$, then the optimal $x = 0$. To see this, note that

$$\frac{\partial W_A}{\partial x} = -\alpha c_0 [1 - \Phi(\frac{s + x + R_A - U}{\sigma_e})]. \qquad \text{(B-91)}$$

This expression is monotonically increasing in x. Also, it converges to $-\alpha c_0$ as $x \to -\infty$, and for $x = -s - R_A + U$, it takes the value of $-\alpha c_0/2$. Since $-\alpha c_0 < \beta < -\alpha c_0/2$, the optimal choice of x takes the form described earlier with s_A defined as the unique solution to $\partial W_A(s_A, 0, \sigma_e, R_A)/\partial x = -\beta$.

Therefore

$$\begin{aligned}
&\frac{d}{d\rho} V(R_A, \sigma_s, \sigma_e) - \frac{d}{d\rho} V(0, \sigma_s, \sigma_e) \\
&< \rho\sigma^2 [\Phi(\frac{s_A}{\sigma_s}) \frac{\partial^2 W_A(s_A, 0, \sigma_e, R_A)}{\partial s^2} - \Phi(\frac{s_1}{\sigma_s}) \frac{\partial^2 W(s_1, 0, \sigma_e)}{\partial s^2} - (\Phi(\frac{s_3}{\sigma_s}) - \Phi(\frac{s_2}{\sigma_s})) \frac{\partial^2 W(s_2, 0, \sigma_e)}{\partial s^2}]. \qquad \text{(B-92)}
\end{aligned}$$

In the rest of the proof, we numerically show the following:

$$\Phi(\frac{s_A}{\sigma_s}) \frac{\partial^2 W_A(s_A, 0, \sigma_e, R_A)}{\partial s^2} - \Phi(\frac{s_1}{\sigma_s}) \frac{\partial^2 W(s_1, 0, \sigma_e)}{\partial s^2} < 0. \qquad \text{(B-93)}$$

Note that $\partial^2 W(s_2, 0, \sigma_e)/\partial s^2 > 0$ given that s_2 is the smaller of the two solutions to $\partial W(s, 0, \sigma_e)/\partial s = \beta$ within $0 \leq s \leq U$, over which $\partial W(s, 0, \sigma_e)/\partial s$ is concave in s. Therefore, proving the preceding inequality is sufficient to prove the proposition.

Notice that if there were at most one solution to the equation defining s_2, implying $x = 0$ for all $s \geq 0$ in the scarce reserve case, then it would still be sufficient to prove the preceding inequality; in that case $d/d\rho(V(R_A, \sigma_s, \sigma_e) - V(0, \sigma_s, \sigma_e))$ would be equal to the left-hand size of the inequality.

Note that

$$\frac{\partial^2 W_A(s_A, 0, \sigma_e, R_A)}{\partial s^2} = \alpha c_0 \mu_e(s_A + R_A - U); \text{ and} \tag{B-94}$$

$$\frac{\partial^2 W(s_1, 0, \sigma_e)}{\partial s^2} = \alpha c_0[2\mu_e(s_1) - \mu_e(s_1 - U)]. \tag{B-95}$$

Thus it is sufficient to show

$$\Phi(\frac{s_A}{\sigma_s})\mu_e(s_A + R_A - U) - \Phi(\frac{s_1}{\sigma_s})[2\mu_e(s_1) - \mu_e(s_1 - U)] < 0. \tag{B-96}$$

First we show that $s_A < s_1$. Note that R_A satisfies the following FOC:

$$\frac{\partial V(R_A, \sigma_s, \sigma_e)}{\partial R} = 0, \tag{B-97}$$

where R is the first argument of V. Given the expression for V and using the envelope property on x, we have

$$\frac{\partial V(R_A, \sigma_s, \sigma_e)}{\partial R} = \int_{-\infty}^{\infty} -\alpha c_0[1 - \Phi(\frac{s + x + R_A - U}{\sigma_e})]\mu_s ds + 1. \tag{B-98}$$

The integrand has the same form as $\partial W/\partial x$ because x and R show up in the form of $s + x + R_A - U$ in the expression for W. If $s \geq s_A$, then $x = 0$ and

$$-\alpha c_0[1 - \Phi(\frac{s + x + R_A - U}{\sigma_e})] = -\alpha c_0[1 - \Phi(\frac{s + R_A - U}{\sigma_e})] < -\beta[1 - \Phi(\frac{s + R_A - U}{\sigma_e})]. \tag{B-99}$$

If $s \leq s_A$, then $x = s_A - s$ and

$$-\alpha c_0[1-\Phi(\frac{s + x + R_A - U}{\sigma_e})] = -\alpha c_0[1-\Phi(\frac{s_A + R_A - U}{\sigma_e})] = -\beta < -\beta[1-\Phi(\frac{s + R_A - U}{\sigma_e})]. \tag{B-100}$$

Thus the FOC for R_A implies

$$\int_{-\infty}^{\infty} -\beta[1 - \Phi(\frac{s + R_A - U}{\sigma_e})]\mu_s ds + 1 > 0. \tag{B-101}$$

Furthermore,

$$\int_{-\infty}^{\infty} [1 - \Phi(\frac{s + R_A - U}{\sigma_e})]\mu_s ds = \int_{-\infty}^{\infty} \Phi(\frac{-s - R_A + U}{\sigma_e})]\mu_s ds$$

$$= Prob(s + e \leq -R_A + U) = \Phi(\frac{-R_A + U}{\sigma}). \tag{B-102}$$

Thus the FOC for R_A implies

$$-R_A + U < \sigma\Phi^{-1}(\frac{1}{\beta}). \tag{B-103}$$

We use this inequality to get an upper bound on $\mu_e(s_A)$:

$$\begin{aligned}
\mu_e(s_A) &= \mu_e(s_A + (R_A - U) - (R_A - U)) \\
&= \frac{1}{\sqrt{2\pi}\rho\sigma} exp(-\frac{1}{2\rho^2\sigma^2}[(s_A + R_A - U)^2 + 2(s_A + R_A - U)(-R_A + U) + (-R_A + U)^2] \\
&< \frac{1}{\sqrt{2\pi}\rho\sigma} exp(-\frac{1}{2\rho^2\sigma^2}(s_A + R_A - U)^2) exp(-\frac{1}{2\rho^2\sigma^2}(-R_A + U)^2) \\
&= \mu_e(s_A + R_A - U) exp(-\frac{1}{2\rho^2\sigma^2}(-R_A + U)^2). \qquad \text{(B-104)}
\end{aligned}$$

The third line follows from the fact that $s_A + R_A - U < 0$ because by definition, $1 - \Phi((s_A + R_A - U)/\sigma_e) = \beta/\alpha c_0 > 1/2$. From assumption A1,

$$(-R_A + U)^2 > \sigma^2(\Phi^{-1}(\frac{1}{\beta}))^2 > \sigma^2 2log2. \tag{B-105}$$

Therefore

$$exp(-\frac{1}{2\rho^2\sigma^2}(-R_A + U)^2) < exp(-log2) = \frac{1}{2}. \tag{B-106}$$

This implies

$$\mu_e(s_A) < \frac{1}{2}\mu_e(s_A + R_A - U). \tag{B-107}$$

The expressions defining s_1 and s_A in terms of $\partial W/\partial s$ and $\partial W_A/\partial s$ imply

$$\Phi(\frac{s_A + R_A - U}{\sigma_e}) = 1 - \frac{\beta}{\alpha c_0}; \text{ and} \tag{B-108}$$

$$\Phi(\frac{s_1}{\sigma_e}) = \frac{1}{2}[1 - \frac{\beta}{\alpha c_0} + \Phi(\frac{s_1}{\sigma_e} - \frac{U}{\sigma_e})] = \frac{1}{2}[\Phi(\frac{s_A + R_A - U}{\sigma_e}) + \Phi(\frac{s_1}{\sigma_e} - \frac{U}{\sigma_e})]. \tag{B-109}$$

Note that $0 < \Phi(s_1/\sigma_e - U/\sigma_e) < \Phi(s_1/\sigma_e)$, which implies

$$\frac{1}{2}\Phi(\frac{s_A + R_A - U}{\sigma_e}) < \Phi(\frac{s_1}{\sigma_e}) < \Phi(\frac{s_A + R_A - U}{\sigma_e}). \tag{B-110}$$

The preceding inequality implies

$$\mu_e(s_1) > \frac{1}{2}\mu_e(s_A + R_A - U). \tag{B-111}$$

To see this, assume that $\mu_e(s_1) \leq (1/2)\mu_e(s_A + R_A - U)$. Notice that

$$\Phi(\frac{s_1}{\sigma_e}) = \int_{-\infty}^{s_1} \mu_e(e)de = \int_{-\infty}^{s_A+R_A-U} \mu_e(e + (s_1 - s_A - R_A + U))de. \tag{B-112}$$

Since $\Phi(s_1/\sigma_e) < \Phi((s_A + R_A - U)/\sigma_e) < 1/2$, $s_1 < s_A + R_A - U < 0$. Therefore, for any $e < s_A + R_A - U$,

$$\begin{aligned}
\mu_e(e + (s_1 - s_A - R_A + U)) =& \mu_e(e)exp(-\frac{1}{2\rho^2\sigma^2}[(s_1 - s_A - R_A + U)^2 \\
&+ 2e(s_1 - s_A - R_A + U)]) \\
<& \mu_e(e)exp(-\frac{1}{2\rho^2\sigma^2}[(s_1 - s_A - R_A + U)^2 \\
&+ 2(s_A + R_A - U)(s_1 - s_A - R_A + U)^2]) \\
&= \mu_e(e)exp(-\frac{1}{2\rho^2\sigma^2}[s_1^2 - (s_A + R_A - U)^2]) \\
&= \mu_e(e)\frac{\mu(s_1)}{\mu(s_A + R_A - U)} \leq \frac{1}{2}\mu_e(e).
\end{aligned} \tag{B-113}$$

Therefore,

$$\begin{aligned}
\Phi(\frac{s_1}{\sigma_e}) =& \int_{-\infty}^{s_A+R_A-U} \mu_e(e + (s_1 - s_A - R_A + U))de \\
<& \int_{-\infty}^{s_A+R_A-U} \frac{1}{2}\mu_e(e)de = \frac{1}{2}\Phi(\frac{s_A + R_A - U}{\sigma_e}).
\end{aligned} \tag{B-114}$$

This is a contradiction. Therefore,

$$\mu_e(s_1) > \frac{1}{2}\mu_e(s_A + R_A - U) > \mu_e(s_A). \tag{B-115}$$

This implies $s_A < s_1$ because both s_A and s_1 are negative. Therefore, $\Phi(s_A/\sigma_s) < \Phi(s_1/\sigma_s)$. Recall that we only need to prove

$$\Phi(\frac{s_A}{\sigma_s})\mu_e(s_A + R_A - U) - \Phi(\frac{s_1}{\sigma_s})[2\mu_e(s_1) - \mu_e(s_1 - U)] < 0. \tag{B-116}$$

Given $\Phi(s_A/\sigma_s) < \Phi(s_1/\sigma_s)$, it is sufficient to prove

$$\mu_e(s_A + R_A - U) - [2\mu_e(s_1) - \mu_e(s_1 - U)] \leq 0. \tag{B-117}$$

Notice that

$$\int_{-\infty}^{0} [2\mu_e(s_1 + x) - \mu_e(s_1 - U - x)]dx = 2\Phi(\frac{s_1}{\sigma_e}) - \Phi(\frac{s_1}{\sigma_e} - \frac{U}{\sigma_e}) = 1 - \frac{\beta}{\alpha c_0}. \tag{B-118}$$

Similarly

$$\int_{-\infty}^{0} \mu_e(s_A + R_A - U + x)dx = 1 - \frac{\beta}{\alpha c_0}. \tag{B-119}$$

These two equations imply

$$\int_{-\infty}^{0} \frac{2\mu_e(s_1 + x) - \mu_e(s_1 - U - x)}{\mu_e(s_A + R_A - U - x)} \mu_e(S_A + R_A - U - x)dx = \int_{-\infty}^{0} \mu_e(s_A + R_A - U + x)dx. \tag{B-120}$$

If we show that, for all $x \leq 0$

$$\frac{2\mu_e(s_1 + x) - \mu_e(s_1 - U - x)}{\mu_e(s_A + R_A - U - x)} \leq \frac{2\mu_e(s_1) - \mu_e(s_1 - U)}{\mu_e(s_A + R_A - U)}, \tag{B-121}$$

then the preceding equation implies $(2\mu_e(s_1) - \mu_e(s_1 - U))/\mu_e(s_A + R_A - U) \geq 1$, which completes the proof.

Note that

$$\begin{aligned}\frac{d}{dx} \frac{2\mu_e(s_1 + x) - \mu_e(s_1 - U + x)}{\mu_e(s_A + R_A - U + x)} &= \frac{1}{\rho^2\sigma^2} \frac{1}{\mu_e(s_A + R_A - U + x)} \\ &\times [2(s_A + R_A - U - s_1)\mu_e(s_1 + x) - (s_A + R_A - s_1)\mu_e(s_1 - U + x)].\end{aligned} \tag{B-122}$$

Next we show that if the following inequality holds for $x = 0$, then it holds for all $x \leq 0$:

$$2(s_A + R_A - U - s_1)\mu_e(s_1 + x) - (s_A + R_A - s_1)\mu_e(s_1 - U + x) \geq 0. \tag{B-123}$$

This is because for all $x \leq 0$,

$$\begin{aligned}\frac{\mu_e(s_1 - U + x)}{\mu_e(s_1 - U)} &= exp(-\frac{1}{2\rho^2\sigma^2}(x^2 + 2x(s_1 - U))) \\ &\leq exp(-\frac{1}{2\rho^2\sigma^2}(x^2 + 2xs_1)) = \frac{\mu_e(s_1 + x)}{\mu_e(s_1)}.\end{aligned} \tag{B-124}$$

Therefore we only need to show

$$2(s_A + R_A - U - s_1)\mu_e(s_1) - (s_A + R_A - s_1)\mu_e(s_1 - U) \geq 0, \tag{B-125}$$

or equivalently,

$$\begin{aligned}&2 + (-1 - \frac{U}{s_A + R_A - U - s_1})exp(-\frac{1}{2\rho^2\sigma^2}(U^2 - 2Us_1)) \\ &= 2 + (-1 - \frac{U}{C_1 - s_1})C_2 \geq 0,\end{aligned} \tag{B-126}$$

where $C_1 \equiv s_A + R_A - U$ and C_2 denotes the expression with *exp* for convenience. This inequality is obtained by dividing the previous inequality by $(s_A + R_A - U - s_1)\mu_e(s_1)$, which is positive.

From assumptions A1 and A2,

$$U > \sigma\Phi^{-1}(1 - \frac{2}{3\alpha c_0}) > \sigma\Phi^{-1}(1 - \frac{2}{3.3\beta}) > 1.45\sigma. \tag{B-127}$$

Differentiating the left-hand side of the inequality B-126 with respect to U, excluding U inside C_1 but including U inside C_2, we obtain

$$\frac{C_2}{C_1 - s_1}[-1 + \frac{U(U - s_1)}{\rho^2\sigma^2}] + \frac{U - s_1}{\rho^2\sigma^2}C_2. \tag{B-128}$$

The expression is positive for any $U > \rho\sigma$ because $s_1 < 0$. Therefore it is sufficient to evaluate the inequality B-126 at $U = 1.45\rho\sigma$, except for the occurrence of U inside C_1, to show that the inequality holds at the actual value of U.

Next we calculate a lower bound on $C_1 - s_1$. Note that

$$\begin{aligned}\Phi(\frac{s_1}{\sigma_e} - \frac{U}{\sigma_e}) = \Phi(\frac{s_1}{\sigma_e})exp(-\frac{U^2 - 2Us_1}{2\rho^2\sigma^2}) &< \Phi(\frac{s_1}{\sigma_e})exp(-\frac{U^2}{2\rho^2\sigma^2}) \\ &< \Phi(\frac{s_1}{\sigma_e})exp(-\frac{1.45^2}{2}).\end{aligned} \tag{B-129}$$

Therefore, based on the equation defining s_1,

$$\Phi(\frac{s_1}{\sigma_e}) < \frac{1}{2}[1 - \frac{\beta}{\alpha c_0} + \Phi(\frac{s_1}{\sigma_e})exp(-\frac{1.45^2}{2})]. \tag{B-130}$$

This inequality implies

$$\Phi(\frac{s_1}{\sigma_e}) < [1 - \frac{1}{2}exp(-\frac{1.45^2}{2})]^{-1}\frac{1}{2}(1 - \frac{\beta}{\alpha c_0}) < 0.61(1 - \frac{\beta}{\alpha c_0}). \tag{B-131}$$

Therefore,

$$C_1 - s_1 > \rho\sigma[\Phi^{-1}(1 - \frac{\beta}{\alpha c_0}) - \Phi^{-1}(0.61(1 - \frac{\beta}{\alpha c_0}))] \tag{B-132}$$

Next we use the following property of a normal distribution: if $a < 1/2$ and $0 < b < 1$, then $(\partial/\partial a)(\Phi^{-1}(a) - \Phi^{-1}(ab)) > 0$. To see this,

$$\frac{\partial}{\partial a}(\Phi^{-1}(a) - \Phi^{-1}(ab)) = \frac{1}{\phi(\Phi^{-1}(a))} - \frac{b}{\phi(\Phi^{-1}(ab))}. \tag{B-133}$$

Using an argument similar to the one used in proving $\mu_e(s_1) > (1/2)\mu_e(s_A + R_A - U)$, we can prove that this expression is positive.

Applying this property to $C_1 - s_1$ and using assumption A2, we obtain the following inequality:

$$\begin{aligned} C_1 - s_1 &> \rho\sigma[\Phi^{-1}(1 - \frac{\beta}{\alpha c_0}) - \Phi^{-1}(0.61(1 - \frac{\beta}{\alpha c_0}))] \\ &> \rho\sigma[\Phi^{-1}(1 - \frac{1}{1.1}) - \Phi^{-1}(0.61(1 - \frac{1}{1.1}))] > 0.25\rho\sigma. \end{aligned} \quad \text{(B-134)}$$

In addition, note that

$$s_1 < \rho\sigma\Phi^{-1}(0.61(1 - \frac{\beta}{\alpha c_0})) < \rho\sigma\Phi^{-1}(0.61(1 - \frac{1}{2})) < -0.51\rho\sigma. \quad \text{(B-135)}$$

Going back to inequality B-126, it is sufficient to prove

$$2 + (-1 - \frac{1.45\rho\sigma}{C_1 - s_1})C_2 \geq 0, \quad \text{(B-136)}$$

with C_2 also evaluated at $U = 1.45\rho\sigma$:

$$C_2 = exp(-\frac{U^2 - 2Us_1}{2\rho^2\sigma^2}) < exp(-\frac{1}{2}1.45(1.45 + 1.02)) < 0.17. \quad \text{(B-137)}$$

Thus, evaluated at $U = 1.45\rho\sigma$,

$$2 + (-1 - \frac{1.45\rho\sigma}{C_1 - s_1})C_2 > 2 + (-1 - \frac{1.45\rho\sigma}{C_1 - s_1}) \times 0.17 > 2 + (-1 - \frac{1.45}{0.25}) \times 0.17 > 0.84. \quad \text{(B-138)}$$

This completes the proof.

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